Use of transverse foliations to the study of area preserving homeomorphisms of surfaces

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Thèse de doctorat

Discipline : Mathématique

présentée par

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Utilisation de feuilletages transverse à l'étude d'homéomorphismes préservant l'aire de surfaces

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Soutenue le 2 décembre 2014 devant le jury composé de :

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Cette thèse est dédiée à mes parents et mon fiancé.

Remerciements

J'aimerais tout d'abord remercier mon directeur de thèse Patrice Le Calvez pour toutes ses aides. Cette thèse n'aurait jamais vu le jour sans ses suggestions pertinentes, sa générosité à partager ses connaissances et idées mathématiques, sa patience pendant les discussions et ses annotations de plusieurs versions de cette thèse. C'est une grande chance et un honneur de pouvoir faire mes etudes et travaux de recherche en mathématique avec lui.

Je remercie ensuite François Béguin et Shigenori Matsumoto d'avoir accepté d'être rapporteurs de cette thèse. Leurs annotations m'ont été précieuses. Je remercie aussi Sylvain Crovisier et Frédéric Le Roux pour ses plusieurs commentaires et m'expliquer ses résultats récents. Je remercie sincèrement Alain Chenciner, John Guaschi et Marco Mazzucchelli de m'avoir fait l'honneur de faire partie du jury.

Mes remerciements vont également à toute l'équipe de l'Institute Mathématique de Jussieu-Paris Rive Gauche où j'ai effectué mon travail de recherche. Je remercie tous mes collègues et amis avec qui j'ai partagé mathématiques et bons moments.

Je tiens particulièrement à remercier Zhiying Wen pour ses encouragements. Enfin, je voudrais exprimer ma sincère gratitude à mes parents et mon fiancé pour leur soutien constant et le bonheur que l'on partage.

Résumé

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Cette thèse concerne les homéomorphismes de surfaces.

Soit f un difféomorphisme d'une surface M préservant l'aire et isotope à l'identité. Si f a un point fixe contractile isolé et dégénéré z_0 avec un indice de Lefschetz égal à 1, et si l'aire de M est finie, nous prouverons au chapitre 3 que z_0 est accumulé non seulement par des points périodiques mais aussi par des orbites périodiques au sens de la mesure. Plus précisément, la mesure de Dirac en z_0 est la limite en topologie faible-étoile d'une suite de probabilités invariantes supportées par des orbites périodiques. Notre preuve est totalement topologique et s'applique au cas d'homéomorphismes en considérant l'ensemble de rotation local.

Au chapitre 4, nous étudierons des homéomorphismes préservant l'aire et isotope à l'identité. Nous prouverons l'existence d'isotopies maximales particulières : les isotopies maximales à torsion faible. En particulier, lorsque f est un difféomorphisme ayant un nombre fini de points fixes tous non-dégénérés, une isotopie I joignant l'identité à f est à torsion faible si et seulement si pour tout point z fixé le long de I, le nombre de rotation (réel) $\rho(I, z)$, qui est bien défini quand on éclate f en z, est contenu dans $(-1, 1)$. Nous démontrerons l'existence d'isotopies maximales à torsion faible, et nous étudierons la dynamique locale de feuilletages transverses à l'isotopie près des singularités isolées.

Au chapitre 5, nous énoncerons une généralisation d'un théorème de Poincaré-Birkhoff local au cas où il existe des points fixes au bord.

Mots-clefs

homéomorphismes de surfaces, feuilletage transverse, orbite périodique, ensemble de rotation local, torsion faible, théorème de Poincaré-Birkhoff

Use of transverse foliations to the study of area preserving homeomorphisms of surfaces

Abstract

This thesis concerns homeomorphisms of surfaces.

Let f be an area preserving diffeomorphism of an oriented surface M isotopic to the identity. If f has an isolated degenerate contractible fixed point z_0 with Lefschetz index one, and if the area of M is finite, we will prove in Chapter 3 that z_0 is accumulated not only by periodic points, but also by periodic orbits in the measure sense. More precisely, the Dirac measure at z_0 is the limit in weak-star topology of a sequence of invariant probability measures supported on periodic orbits. Our proof is purely topological and will works for homeomorphisms and is related to the notion of local rotation set.

In chapter 4, we will define a kind of identity isotopies: torsion-low isotopies. In particular, when f is a diffeomorphism with finitely many fixed points such that every fixed point is not degenerate, an identity isotopy I of f is torsion-low if and only if for every point z fixed along the isotopy, the (real) rotation number $\rho(I, z)$, which is well defined when one blows-up f at z, is contained in $(-1, 1)$. We will prove the existence of torsion-low maximal identity isotopies, and we will deduce the local dynamics of the transverse foliations of any torsion-low maximal isotopy near any isolated singularity.

In chapter 5, we will generalize a local Poincaré-Birkhoff theorem to the case where there exist fixed points on the boundary.

Keywords

homeomorphisms of surfaces, transverse foliation, periodic orbits, local rotation set, torsion-low isotopy, Poincaré-Birkhoff theorem

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Chapter 1

Introduction

1.1 The motives

1.1.1 From Conley's Conjecture to the dynamics near an isolated degenerate fixed point with Lefschetz index 1

A time-dependent vector field $(X_t)_{t\in\mathbb{R}}$ is called a *Hamiltonian vector field* if it is defined by the equation:

$$dH_t = \omega(X_t, \cdot),$$

where (M,ω) is a symplectic manifold, and $H : \mathbb{R} \times M \to \mathbb{R}$ is a smooth function. The Hamiltonian vector field induces a flow $(\varphi_t)_{t\in\mathbb{R}}$ on M, which is the solution of the following equation

$$\frac{\partial}{\partial t}\varphi_t(z) = X_t(\varphi_t(z)).$$

We say that a diffeomorphism f of M is a *Hamiltonian diffeomorphism* if it is the time-1 map of a Hamiltonian flow.

Let us recall some classical problems in symplectic geometry about existence of fixed points or periodic points of Hamiltonian diffeomorphisms of compact symplectic manifolds, which correspond to the periodic solutions of the Hamiltonian equations. A famous conjecture corresponding to the lower bound of the number of fixed points was formulated in 1960s by Vladimir I. Arnold and was proved in different cases. In the case of tori, Arnold's conjecture was proved by Charles Conley and Eduard Zehnder [CZ83]. The most important breakthrough was Andreas Floer's proof [Flo89] in the case of monotone symplectic manifold by introducing a notion of Floer homology. In the study of Arnold's conjecture, Conley conjectured the existence of infinitely many periodic points for Hamiltonian diffeomorphisms of tori. This conjecture was proved by Nancy Hingston [Hin09], and was generalized to Hamiltonian diffeomorphisms of closed symplectically aspherical manifold by Viktor L. Ginzburg [Gin10]. But in the non-degenerate case, where 1 is not an eigenvalue of the Jacobian matrix at every fixed point of the Hamiltonian diffeomorphism, Conley's conjecture has been proven much earlier (see [CZ86] for tori case, and [SZ92] for more general case). In the case where the symplectic manifold is a surface, the situation is a little bit special because we have particular tools (for example, the Poincaré-Birkhoff theorem), and Conley's conjecture has been proven earlier and is generalized to homeomorphism (see [FH03] and [LC06]). Recall that the Poincaré-Birkhoff theorem states that every orientation and area preserving homeomorphism of an annulus that rotates the two boundaries in opposite directions has at least two fixed points. This theorem has a lot of generalizations (see [Bir26], [Car82], [Fra88], and [Gui94] for example), which are efficient tools to ensures the existence of periodic points in the surface.

In particular, when the symplectic manifold is a closed surface with positive genus and the Hamiltonian diffeomorphism has no degenerate fixed point, Conley's conjecture can be easily deduced. Indeed, if f is a Hamiltonian diffeomorphism of a closed surface M with positive genus that has finitely many fixed points, John Franks [Fra96] proved that f has a contractible fixed point z with Lefschetz index 1. Here *contractible* means that the trajectory of z along the Hamiltonian flow that induces f is homotopic to zero in M, and hence there is an identity isotopy of f that fixes z. If f does not have any degenerate fixed point, we lift f to the universal covering space $\widetilde{M}$ of M and get a homeomorphism $\widetilde{f}$ which fixes all the lifts of z. We take one lift $\widetilde{z}$ of z, and can blow-up $\widetilde{M}$ at $\widetilde{z}$ by replacing $\widetilde{z}$ with the unit circle S^1. Moreover, $D\widetilde{f}(\widetilde{z})$ induces a continuous extension of $\widetilde{f}$ to S^1, and we get a homeomorphism of the annulus $\widetilde{M} \setminus \{z\} \cup S^1$. We still denote by $\widetilde{f}$ this homeomorphism. By the non-degeneracy of $\widetilde{z}$, the Poincaré's rotation number of $\widetilde{f}|_{S^1}$ is not equal to $0 \in \mathbb{R}/\mathbb{Z}$. By a generalization of Poincaré-Birkhoff Theorem, one deduces that $\widetilde{f}$ has infinitely many periodic points that correspond to different periodic points of f.

In the degenerate case, the problem is much more difficult. As in the previous paragraph, there always exists a contractible fixed point z with Lefschetz index 1 for a Hamiltonian diffeomorphism of closed surface with finitely many fixed points. In chapter 3, we will study this case and prove that if z is an isolated degenerate contractible fixed point of f with Lefschetz index 1, then z is accumulated by periodic points of f. This gives a new explanation of Conley's Conjecture for Hamiltonian diffeomorphism of closed surface with positive genus.

More generally, we do not really need f to be a Hamiltonian diffeomorphism. Instead, our proof is purely topological. We can define the *Hamiltonian homeomorphisms* of surfaces to be area preserving homeomorphisms that are isotopic to the identity and have zero mean rotation vector. Here, *area preserving* means that f preserves a Borel measure without atom such that the measure of each open set is positive and that the measure of each compact set is finite. Due to Shigenori Matsumoto [Mat01], Franks' result about the existence of a fixed point with Lefschez index 1 was generalized to the case of Hamiltonian homeomorphism. The rotation number can also be generalized in the case of homeomorphism to a notion of the *local rotation set* introduced by Frédéric Le Roux [LR13]. By a similar but more complicate discussion, we can give a new explanation of Conley's conjecture for Hamiltonian homeomorphism of closed surface with positive genus.

More precisely, we will study the case where $f : M \to M$ is an area preserving homeomorphism of an oriented surface M, I is an isotopy from the identity to f, and z_0 is an isolated fixed point of f with a Lefschetz index 1 which is also fixed by I and satisfies $\rho_s(I, z_0) = \{k\}$, where $\rho_s(I, z_0)$ is the local rotation set (see Section 1.2.1). In particular, when f is a diffeomorphism, the condition $\rho_s(I, z_0) = \{k\}$ means that $Df(z_0)$ has a positive real eigenvalue. Under these assumptions, Le Roux [LR13] conjectured that z_0 is accumulated by periodic orbits. We will approach his conjecture by proving that if the total area of M is finite, then z_0 is accumulated not only by periodic points, but also by periodic orbits in the measure sense. More precisely, the Dirac measure at z_0 is the limit in the weak-star topology of a sequence of invariant probability measures supported on periodic orbits.

1.1.2 Searching for transverse foliations

In the proof of the first problem, we will use transverse foliations. More precisely, suppose that there exist (non-singular) oriented topological foliations on M, and fix such a foliation $\mathcal{F}$. We say that a path $\gamma : [0,1] \to M$ is positively transverse to $\mathcal{F}$ if it locally meets transversely every leaf from the left to the right. We say that $\mathcal{F}$ is a *transverse foliation* of an isotopy I, if for every $z \in M$, there exists a path that is homotopic to the trajectory of z along I and is positively transverse to $\mathcal{F}$. Denote by $\mathrm{Fix}(I)$ the set of fixed points of f that is also fixed along I. When $\mathrm{Fix}(I)$ is not empty, we call $\mathcal{F}$ a (singular) *transverse foliation* of I, if $\mathrm{Fix}(I)$ is the set of singularities of $\mathcal{F}$ and if the restriction of $\mathcal{F}$ to $M \setminus \mathrm{Fix}(I)$ is a transverse foliation of the restriction of I to $M \setminus \mathrm{Fix}(I)$.

Transverse foliations are fruitful tools in the study of homeomorphisms of surfaces. For example, one can prove the existence of periodic orbits in several cases [LC05], [LC06]; one can give precise descriptions of the dynamics of some homeomorphisms of the torus $\mathbb{R}^2/\mathbb{Z}^2$ [Dáv13], [KT14]; ...It is a natural question whether we can get a more efficient tool by choosing suitable maximal identity isotopies.

Of course the existence of a transverse foliation prohibits the existence of fixed points of I but also contractible fixed points of f associated to I. Patrice Le Calvez [LC05] proved that if f does not have any contractible fixed point associated to I, there exists a transverse foliation of I. Later, Olivier Jaulent [Jau14] generalized this result to the case where there exist contractible fixed points, and obtained singular foliations. Denote by $\mathrm{Fix}(f)$ the set of fixed points of f. Jaulent proved that there exist a closed subset $X \subset \mathrm{Fix}(f)$ and an identity isotopy I_X on $M \setminus X$ such that $f|_{M\setminus X}$ does not have any contractible fixed point associated to I_X. It means that there exists a singular foliation on M whose set of singularities is X and whose restriction to $M \setminus X$ is transverse to I_X. Recently, François Béguin, Sylvain Crovisier, and Le Roux [BCLR] generalized Jaulent's result, and proved that there exists an identity isotopy I of f such that $f|_{M\setminus \mathrm{Fix}(I)}$ does not have any contractible fixed point associated to $I|_{M\setminus \mathrm{Fix}(I)}$. Then, there exists a singular foliation on M whose set of singularities is the set of fixed points of I and whose restriction to $M \setminus \mathrm{Fix}(I)$ is transverse to $I|_{M\setminus \mathrm{Fix}(I)}$. We call such an identity isotopy I a *maximal identity isotopy.*

The primary idea is to choose a maximal isotopy that fixes as many fixed points as possible. When $f : M \to M$ is an orientation preserving diffeomorphism, and I is an identity isotopy of f fixing z_0, we can give a natural blow-up at z_0 and extend f continuously to the circle added as in the previous section. Moreover, we can define a *blow-up rotation number* $\rho(I, z_0) \in \mathbb{R}$, that is a representative of the Poincaré's rotation number on the circle added (see Section 2.10). Moreover, if the diffeomorphism f is area preserving, and if there exists a fixed point $z_0 \in \mathrm{Fix}(I)$ such that $|\rho(I, z_0)| > 1$ and that the connected component M_0 of $M \setminus (\mathrm{Fix}(I) \setminus \{z_0\})$ containing z_0 is not homeomorphic to a sphere or a plane, by lift $f|_{M\setminus(\mathrm{Fix}(I)\setminus\{z_0\})}$ to the universal cover of $M \setminus (\mathrm{Fix}(I) \setminus \{z_0\})$, we can find another fixed point of f that is not a fixed point of I as a corollary of a generalized version of Poincaré-Birkhoff theorem. Moreover, if $\mathrm{Fix}(I)$ is finite, by a technical construction, one can find another identity isotopy that fixes $\mathrm{Fix}(I) \setminus \{z_0\}$ and has no less (probably more) fixed points than I (see Section 4.2). Then, it is reasonable to think that a maximal identity isotopy I such that

$$-1 \leq \rho(I, z) \leq 1 \text{ for all } z \in \mathrm{Fix}(I),$$

fixes more fixed points than a usual one. In chapter 4, we will study a more general case, and prove the existence of such an isotopy as a corollary.

More precisely, we will study area preserving homeomorphisms of an oriented surface isotopic to the identity, and prove the existence of a special kind of maximal identity isotopies: the torsion-low maximal identity isotopies. In this case, we also have more information about its transverse foliation: we can deduce the local dynamics of a transverse foliation near any isolated singularity.

1.1.3 Generalized local Poincaré-Birkhoff theorem to the case where there exist fixed points at the boundary

We are also interested in the generalization of local Poincaré-Birkhoff theorem. More precisely, let f be an orientation preserving homeomorphism of the annulus $\mathbb{T}^1 \times [0, +\infty)$. Let $C_0 = \mathbb{T}^1 \times \{0\}$, C_1 be an essential loop in $\mathbb{T}^1 \times (0, +\infty)$ that projects injectively to the first factor, and $C_2 = f(C_1)$. When f twists the boundaries and satisfies an intersection condition, it has been proven that f has a fixed point in the annulus between C_0 and C_1 (see [Bir26] and [Gui94] for example). But what would happen if there exists a fixed point of f in C_0? We will give an answer in Chapter 5. Moreover, this result can be used to give a more delicate description of the dynamics of some local homeomorphisms.

1.2 Statements of the precise results of the thesis

1.2.1 The results of Chapter 3

Let $f : M \to M$ be an area preserving homeomorphism of an oriented surface M, z_0 be an isolated fixed point of f, and I be an identity isotopy of f fixing z_0. For two small neighborhoods $V \subset U$ of z_0, we will write

$$\rho_{U,V}(I) = \cap_{m \geq 1} \overline{\cup_{n \geq m} \{\rho_n(z) : z \in \cap_{0 \leq j \leq n} f^{-j}(U) \setminus (V \cup f^{-n}(V))\}},$$

where $\rho_n(z)$ is the average change of angular coordinate along the trajectory of z. We define the *local rotation set* to be

$$\rho_s(I, z_0) = \cap_U \overline{\cup_V \rho_{U,V}(I)}.$$

Let us say that a contractible q-periodic orbit has *type* (p, q) associated to I at z_0 if its trajectory along I is homotopic to $p\Gamma$ in $M \setminus \mathrm{Fix}(I)$, where Γ is the boundary of a sufficiently small Jordan domain containing z_0. We will prove the following result which is the main result of Chapter 3.

Theorem 1.1. *Suppose that the Lefschetz index $i(f, z_0)$ is equal to 1, and that the rotation set $\rho_s(I, z_0)$ is reduced to an integer. If one of the following situations occurs,*

i) the surface M is a plane, f has exactly one fixed point and has a periodic orbit besides z_0;

ii) the total area of M is finite,

then z_0 is accumulated by periodic points. More precisely, the following property holds:

***P)**: There exists $\varepsilon > 0$, such that either for all irreducible $p/q \in (k, k + \varepsilon)$, or for all irreducible $p/q \in (k - \varepsilon, k)$, there exists a contractible periodic orbit $O_{p/q}$ of type (p, q), such that $\mu_{O_{p/q}} \to \delta_{z_0}$ as $p/q \to k$, in the weak-star topology, where $\mu_{O_{p/q}}$ is the invariant probability measure supported on $O_{p/q}$,*

The rotation set of a local isotopy at a degenerate fixed point of an orientation and area preserving diffeomorphism is reduced to an integer. So, given an area preserving diffeomorphism f isotopic to the identity on a surface M with finite area, if z_0 is a degenerate fixed point whose Lefschetz index is equal to 1, the assumptions of the previous theorem are satisfied, and hence z_0 is accumulated by contractible periodic points. Formally, we have the following corollary:

Corollary 1.2. *Let f be an area preserving diffeomorphism of an oriented surface M with finite total area, and z_0 be a degenerate isolated fixed point such that $i(f, z_0) = 1$. If f is isotopic to the identity by an isotopy I that fixes z_0, then z_0 is accumulated by contractible periodic points. Moreover, the property **P)** holds.*

Let f be a $\mathcal{C}^1$ diffeomorphism of $\mathbb{R}^2$. A function $g : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^2$ is called a *generating function* of f if $\partial^2_{12} g < 1$, and

$$f(x,y) = (X,Y) \Leftrightarrow \begin{cases} X - x = & \partial_2 g(X,y), \\ Y - y = & -\partial_1 g(X,y). \end{cases}$$

We know that the previous diffeomorphism f is orientation and area preserving by a direct computation.

Generating functions are usual objects in symplectic geometry. We will give the following version of our result whose conditions are described by generating functions.

Corollary 1.3. *Let f be an area preserving diffeomorphism of an oriented surface M with finite area that is isotopic to the identity by an isotopy fixing $z_0 \in M$. Suppose that in a neighborhood of z_0, f is conjugate to a local diffeomorphism at 0 that is generated by a generating function g, that 0 is a local extremum of g, and that the Hessian matrix of g at 0 is degenerate. Then z_0 is accumulated by periodic points, and the property **P)** holds.*

In particular, a Hamiltonian diffeomorphism f of the torus $\mathbb{T}^2$ that is close to the identity in $\mathcal{C}^1$ topology can be lifted to the plane $\mathbb{R}^2$, and the lifted diffeomorphism can be defined by a generating function g. If z_0 is a local extremum of g, and if the Hessian of g at z_0 is degenerate, we are in the case of the previous corollary, and the image of z_0 in $\mathbb{T}^2$ is a fixed point of f that is accumulated by contractible periodic points.

We will also give a version of our result whose assumptions are described by symplectically degenerate extremum that will be defined in the section 3.2.2. Marco Mazzucchelli noticed that the existence of a symplectically degenerate extremum implies the existence of infinitely many other periodic points, and asked whether a symplectically degenerate extremum actually corresponds to a fixed point accumulated by periodic points, in his paper [Maz13] which gave a simpler proof of Conley's conjecture of tori. In chapter 3, we will give a positive answer of Mazzucchelli's question in the case where the torus is the surface $\mathbb{T}^2$. More precisely, we have the following result.

Theorem 1.4. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a Hamiltonian diffeomorphism, and z_0 be a symplectically degenerate extremum. Then z_0 is accumulated by periodic points, and the property **P)** holds.*

1.2.2 The results of Chapter 4

We write $f : (W, 0) \to (W', 0)$ for an orientation preserving homeomorphism between two neighborhoods W and W' of 0 in $\mathbb{R}^2$ such that $f(0) = 0$. We say that f is an *orientation preserving local homeomorphism at* 0. More generally, we write $f : (W, z_0) \to (W', z_0)$ for an orientation preserving local homeomorphism between two neighborhoods W and W' of z_0 in any oriented surface M such that $f(z_0) = z_0$.

Let $f:(W,z_0)\to(W',z_0)$ be an orientation preserving local homeomorphism at z_0. A *local isotopy* I of f is a continuous family of local homeomorphisms $(f_t)_{t\in[0,1]}$ fixing z_0. Suppose that f is not conjugate to a contraction or an expansion. We can give a preorder on the space of local isotopies such that for two local isotopies I and I', one has $I\lesssim I'$ if and only if there exists $k\geq 0$ such that I' is locally homotopic to $J_{z_0}^k I$, where $J_{z_0}=(R_{2\pi t})_{t\in[0,1]}$ is the local isotopy of the identity such that each $R_{2\pi t}$ is the counter-clockwise rotation through an angle $2\pi t$ about the center z_0. We will give the formal definitions in Section 2.3.

Let $\mathcal{F}$ be a singular oriented foliation on M. We say that $\mathcal{F}$ is *locally transverse* to a local isotopy $I=(f_t)_{t\in[0,1]}$ at z_0, if there exists a neighborhood U_0 of z_0 such that $\mathcal{F}|_{U_0}$ has exactly one singularity z_0, and if for every sufficiently small neighborhood U of z_0, there exists a neighborhood $V\subset U$ such that for all $z\in V\setminus\{z_0\}$, there exists a path in $U\setminus\{z_0\}$ that is homotopic in $U\setminus\{z_0\}$ to the trajectory $t\mapsto f_t(z)$ of z along I and is positively transverse to $\mathcal{F}$.

We will generalize the definitions of "positive type" and "negative type" by Matsumoto [Mat01]. We say that I has a *positive (resp. negative) rotation type* at z_0 if there exists a foliation $\mathcal{F}$ locally transverse to I such that z_0 is a sink (resp. source) of $\mathcal{F}$. We say that I has a *zero rotation type* at z_0 if there exists a foliation $\mathcal{F}$ locally transverse to I such that z_0 is an isolated singularity of $\mathcal{F}$ and is neither a sink nor a source of $\mathcal{F}$. Two local isotopies I and I' have the same rotation type if they are equivalent.

When z_0 is an isolated fixed point of f, a local isotopy of f has at least one of the previous rotation types. It is possible that a local isotopy of f has two rotation types. Let us say that f is *locally non-wandering* if there exists a neighborhood of z_0 that contains neither the positive nor the negative orbit of any wandering open set. If we assume that f is area preserving (or more generally f is locally non-wandering), we will show in Section 4.1 that a local isotopy of f has exactly one of the three rotation types. We say that I is *torsion-low* at z_0 if

- every local isotopy $I'>I$ has a positive rotation type;
- every local isotopy $I'<I$ has a negative rotation type.

Under the previous assumptions, we will prove in Section 4.1 the existence of a torsion-low local isotopy I of f. Formally, we have the following result:

Theorem 1.5. *Let $f:(W,z_0)\to(W',z_0)$ be an orientation preserving local homeomorphism at an isolated fixed point z_0. If f is locally non-wandering, then*

- *a local isotopy of f has exactly one of the three kinds of rotation types;*
- *there exists a local isotopy I_0 that is torsion-low at z_0. Moreover, I_0 has a zero rotation type if the Lefschetz index $i(f,z_0)$ is different from 1, and has either a positive or a negative rotation type if the Lefschetz index $i(f,z_0)$ is equal to 1.*

A torsion-low local isotopy has the following properties:

Proposition 1.6. *Let $f:(W,z_0)\to(W',z_0)$ be an orientation preserving and locally non-wandering homeomorphism at an isolated fixed point z_0, and I be a local isotopy of f. If I is torsion-low at z_0, then*

$$\rho_s(I,z_0)\subset[-1,1].$$

In particular, if f can be blown-up at z_0, the rotation set is reduced to a real number in $[-1,1]$. Moreover, if f is a diffeomorphism in a neighborhood of z_0, the blow-up rotation number satisfies

$$-1\leq\rho(I,z_0)\leq 1,$$

and the inequalities are both strict when z_0 is not degenerate.

When z_0 is not an isolated fixed point and f is area preserving, we will generalize the definition of torsion-low isotopy by considering the local rotation set. We say a local isotopy I of an orientation and area preserving local homeomorphism f at a non-isolated fixed point z_0 is *torsion-low* at z_0 if $\rho_s(I, z_0) \cap [-1, 1] \neq \emptyset$. One may fail to find a torsion-low local isotopy in some particular cases. In fact, there exists an orientation and area preserving local homeomorphism whose local rotation set is reduced to ∞, and hence there does not exist any torsion-low isotopy of this local homeomorphism. We will give such an example in Section 4.3.

However, if f is an area preserving homeomorphism of an oriented surface M that is isotopic to the identity, we can find a maximal identity isotopy I that is torsion-low as a local isotopy at each fixed point of I. Formally, we will prove the following theorem in Section 4.5, which is the main result of the Chapter 4.

Theorem 1.7. *Let f be an area preserving homeomorphism of an oriented surface M that is isotopic to the identity. Then, there exists a maximal identity isotopy I such that I is torsion-low at z for every $z \in \mathrm{Fix}(I)$.*

Remark 1.8. The area preserving condition is necessary for the result of this theorem. Even if f has only finitely many fixed points and is area preserving near each fixed point, one may still fail to find a maximal isotopy I that is torsion-low at every $z \in \mathrm{Fix}(I)$. We will give such an example in Section 4.3.

We say that an identity isotopy is *torsion-low* if it is torsion-low at each of its fixed points. A torsion-low maximal isotopy gives more information than a usual one. We have the following three results:

Proposition 1.9. *Let f be an area preserving homeomorphism of an oriented surface M that is isotopic to the identity and has finitely many fixed points. Let*

$$n = \max\{\#\mathrm{Fix}(I) : I \textit{ is an identity isotopy of } f\}.$$

Then, there exists a torsion-low identity isotopy of f with n fixed points.

Proposition 1.10. *Let f be an area preserving homeomorphism of an oriented surface M that is isotopic to the identity, I be a maximal identity isotopy that is torsion-low at $z \in \mathrm{Fix}(I)$, and $\mathcal{F}$ be a transverse foliation of I. If z is isolated in the set of singularities of $\mathcal{F}$, then we have the following results:*

- *if z is an isolated fixed point of f such that $i(f, z) \neq 1$, then z is a saddle*[1] *of $\mathcal{F}$ and $i(\mathcal{F}, z) = i(f, z)$;*
- *if z is an isolated fixed point such that $i(f, z) = 1$, or if z is not isolated in* $\mathrm{Fix}(f)$, *then z is a sink or a source of $\mathcal{F}$.*

Proposition 1.11. *Let f be an area preserving diffeomorphism of an oriented surface M that is isotopic to the identity. Then, there exists a maximal identity isotopy I, such that for all $z \in \mathrm{Fix}(I)$,*

$$-1 \leq \rho(I, z) \leq 1.$$

Moreover, the inequalities are both strict when z is not degenerate.

1. The precise definitions of a saddle, a sauces and a sink will be given in Section 2.7.

1.2.3 The results of Chapter 5

Let f be an orientation preserving homeomorphism of the annulus $\mathbb{T}^1 \times [0, +\infty)$. An *essential* loop in the annulus is a loop that is not homotopic to zero. Let $C_0 = \mathbb{T}^1 \times \{0\}$, C_1 be an essential loop in $\mathbb{T}^1 \times (0, +\infty)$ that projects injectively to the first factor, and $C_2 = f(C_1)$. We denote by A_i the closed annulus bounded by C_0 and C_i. Let $\pi : \mathbb{R} \times [0, \infty) \to \mathbb{T}^1 \times [0, \infty)$ be the universal cover, and $\widetilde{f}$ be a lift of f. Let $p_1 : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be the projection on the first factor. We denote by $\widetilde{C}_i$ the pre-images of C_i for $i = 0, 1, 2$. We denote by $\mathrm{Fix}_*(f)$ the set of fixed points of f lifted to fixed points of $\widetilde{f}$.

We will prove the following results in Chapter 5:

Theorem 1.12. *If $\mathrm{Fix}_*(f) \cap C_0 = \{z_0\}$, and if $p_1(\widetilde{f}(\widetilde{z}) - \widetilde{z})p_1(\widetilde{f}(\widetilde{z}') - \widetilde{z}') < 0$ for all $\widetilde{z} \in \widetilde{C}_0 \setminus \pi^{-1}(z_0)$ and $\widetilde{z}' \in \widetilde{C}_1$, then we are in one of the following cases:*
i) there exists a fixed point of f in the interior of A_1 lifted to fixed points of $\widetilde{f}$;
ii) there exists an essential loop γ in A_1 that does not intersect $C_0 \setminus \{z_0\}$ and satisfies $\gamma \cap f(\gamma) \subset \{z_0\}$.

Corollary 1.13. *If A_1 does not contain the positive or the negative orbit of any wandering open set and if there exists $\widetilde{z} \in \widetilde{C}_0$ such that $p_1(\widetilde{f}(\widetilde{z}) - \widetilde{z})p_1(\widetilde{f}(\widetilde{z}') - \widetilde{z}') < 0$ for all $\widetilde{z}' \in \widetilde{C}_1$, then there exists a fixed point of f in the interior of A_1 lifted to fixed points of $\widetilde{f}$.*

We will also give a description of the dynamics of some local homeomorphisms as a corollary of our main result. Suppose that $f : (W, z_0) \to (W, z_0)$ is an orientation preserving local homeomorphism at z_0 that is locally non-wandering, and that I is a local isotopy of f that has a positive (resp. negative) rotation type. We know that if f can be blown-up at z_0, the rotation number is included in $[0, \infty]$ (resp. $[-\infty, 0]$) (see [LR13] or Section 2.10). We know also that if z_0 is a non-accumulated indifferent point, then there exist small invariant continuums at z_0. Choosing a sufficiently small invariant continuum K, we can blown-up $f|_{W \setminus K}$, and the rotation number $\rho(I, K)$ is included in $[0, \infty]$ (resp. $[-\infty, 0]$) (see Section 2.9 and 2.10). We say that a homeomorphism h of the circle is *right semi-stable* (resp. *left semi-stable*) if its lift $\widetilde{h}$ to $\mathbb{R}$ satisfies $\widetilde{h}(x) \geq x$ (resp. $\widetilde{h}(x) \leq x$) for all $x \in \mathbb{R}$, and the equality holds at some points. We have the following result:

Corollary 1.14. *In the previous two cases, if $\rho(I, z_0) = 0$, the dynamics on the circle added when blowing-up is right semi-stable (resp. left semi-stable).*

Remark 1.15. In the proof of the first statement of Theorem 1.5, we can also using this corollary instead of Guillou's generalization of Poincaré-Birkhoff theorem.

1.3 Organization of the thesis

Now we give a plan of thesis.

In Chapter 2, we will introduce many definitions and will recall previous results that will be essential in the proofs of our results.

In Chapter 3, we will study the dynamics near an degenerate fixed point with Lefschetz index one. We will first prove our main result of this part: Theorem 1.1; then, we will study a particular case where f is a diffeomorphism, and will proof several version of our results whose conditions are described in different ways: Corollary 1.3 and Theorem 1.4.

In Chapter 4, we will give the existence of torsion-low maximal isotopies and describe its properties. We will first study the local rotation types at an isolated fixed point of an orientation preserving homeomorphism, and will prove Theorem 1.5 and Proposition

1.6; then we will also study the dynamics in global sense and prove the existence of a global torsion-low maximal identity isotopy: Theorem 1.7 in two cases, and will study its properties: Proposition 1.9, 1.10 and 1.11; next, we will give some explicit examples to get the optimality of our results.

In Chapter 5, we will prove two locally generalization of Poincaré-Birkhoff theorem: Theorem 1.12 and Corollary 1.13. We will also using our result to give a description of the dynamics of some local homeomorphisms: Corollary 1.14.

Chapter 2

Preliminaries

2.1 A classification of isolated fixed points

In this section, we will give a classification of isolated fixed points. More details can be found in [LC03].

Let $f : (W, z_0) \to (W', z_0)$ be a local homeomorphism with an isolated fixed point z_0. We say that z_0 is an *accumulated point* if every neighborhood of z_0 contains a periodic orbit besides z_0. Otherwise, we say that z_0 is a *non-accumulated point*.

We define a *Jordan domain* to be a bounded domain whose boundary is a simple closed curve. We say that z_0 is *indifferent* if there exists a neighborhood $V \subset \overline{V} \subset W$ of z_0 such that for every Jordan domain $U \subset V$ containing z_0, the connected component of $\cap_{k\in\mathbb{Z}} f^{-k}(\overline{U})$ containing z_0 intersects the boundary of U.

We say that z_0 is *dissipative* if there exists a fundamental system $\{U_\alpha\}_{\alpha\in J}$ of the neighborhood of z_0 such that each U_α is a Jordan domain and that $f(\partial U_\alpha) \cap \partial U_\alpha = \emptyset$.

We say that z_0 is a *saddle* point if it is neither indifferent nor dissipative.

Note that if f is area preserving, an isolated fixed point of f is either an indifferent point or a saddle point.

2.2 Lefschetz index

Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at an isolated fixed point $0 \in \mathbb{R}^2$. Denote by S^1 the unit circle. If $C \subset W$ is a simple closed curve which contains no fixed point of f, then we can define the *index* $i(f, C)$ of f along the curve C to be the Brouwer degree of the map

$$\begin{aligned} \varphi : S^1 &\to S^1 \\ t &\mapsto \frac{f(\gamma(t)) - \gamma(t)}{||f(\gamma(t)) - \gamma(t)||}, \end{aligned}$$

where $\gamma : S^1 \to C$ is a parametrization compatible with the orientation, and $\|\cdot\|$ is the usual Euclidean norm. Let U be a Jordan domain containing 0 and contained in a sufficiently small neighborhood of 0. We define the *Lefschetz index* of f at 0 to be $i(f, \partial U)$, which is independent of the choice of U. We denote it by $i(f, 0)$.

More generally, if $f : (W, z_0) \to (W', z_0)$ is an orientation preserving local homeomorphism at a fixed point z_0 on a surface M, we can conjugate it topologically to an orientation preserving local homeomorphism g at 0 and define the *Lefschetz index* of f at z_0 to be $i(g, 0)$, which is independent of the choice of the conjugation. We denote it by $i(f, z_0)$.

2.3 Local isotopies and the index of local isotopies

Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at $z_0 \in M$. A *local isotopy* I of f at z_0 is a family of homeomorphisms $(f_t)_{t\in[0,1]}$ such that

- every f_t is a homeomorphism between the neighborhoods $V_t \subset W$ and $V_t' \subset W'$ of z_0, and $f_0 = \mathrm{Id}_{V_0}$, $f_1 = f|_{V_1}$;
- for all t, one has $f_t(z_0) = z_0$;
- the sets $\{(z,t) \in M \times [0,1] : z \in V_t\}$ and $\{(z,t) \in M \times [0,1] : z \in V_t'\}$ are both open in $M \times [0,1]$;
- the maps $(z,t) \mapsto f_t(z)$ and $(z,t) \mapsto f_t^{-1}(z)$ are both continuous.

Let us introduce the index of a local isotopy which was defined by Le Roux [LR13] and Le Calvez [LC08].

Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at $0 \in \mathbb{R}^2$, and $I = (f_t)_{i\in[0,1]}$ be a local isotopy of f. We denote by D_r the disk with radius r and centered at 0. Then the isotopy f_t is well defined in the disk D_r if r is sufficiently small. Let

$$\begin{array}{rcl} \pi : \mathbb{R} \times (-\infty, 0) & \to & \mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\} \\ (\theta, y) & \mapsto & -ye^{i2\pi\theta}, \end{array}$$

be the universal covering projection, and $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ be the lift of $I|_{D_r\setminus\{0\}}$ to $\mathbb{R} \times (-r, 0)$ such that f_0 is the identity. Let $\widetilde{\gamma} : [0,1] \to \mathbb{R} \times (-r, 0)$ be a path from $\widetilde{z'} \in \mathbb{R} \times (-r, 0)$ to $\widetilde{z'} + (1, 0)$. The map

$$t \mapsto \frac{\widetilde{f}_1(\widetilde{\gamma}(t)) - \widetilde{\gamma}(t)}{||\widetilde{f}_1(\widetilde{\gamma}(t)) - \widetilde{\gamma}(t)||}$$

takes the same value at both 0 and 1, and hence descends to a continuous map $\varphi : [0,1]/_{0\sim 1} \to S^1$. We define the *index of the isotopy* I at 0 to be the Brouwer degree of φ, which does not depend on the choice of $\widetilde{\gamma}$ when r is sufficiently small. We denote it by $i(I, 0)$.

For two local isotopies (resp. isotopies) $I = (f_t)_{t\in[0,1]}$ and $I' = (g_t)_{t\in[0,1]}$, we denote by I^{-1} the local isotopy (resp. isotopy) $(f_t^{-1})_{t\in[0,1]}$, by $I'I$ the local isotopy (resp. isotopy) $(\varphi_t)_{t\in[0,1]}$ such that

$$\varphi_t = \begin{cases} f_{2t} & \text{for } t \in [0, \frac{1}{2}], \\ g_{2t-1} \circ f & \text{for } t \in [\frac{1}{2}, 0], \end{cases}$$

and by I^n the local isotopy (resp. isotopy) $\underbrace{I \cdots I}_{n \text{ times}}$ for every $n \geq 1$.

Recall that $\pi_1(\mathrm{homeo}_0(\mathbb{R}^2, 0)) \cong \mathbb{Z}$, where $\mathrm{homeo}_0(\mathbb{R}^2, 0)$ is the space of homeomorphism of $\mathbb{R}^2$ fixing 0 and isotopic to the identity (see [McC63] or [Ham66]). Then, the space of homotopy classes of local isotopies of f is isomorphism to $\mathbb{Z}$. Let $J = (R_{2\pi t})_{t\in[0,1]}$ be the isotopy such that each $R_{2\pi t}$ is the counter-clockwise rotation through an angle $2\pi t$ about the center 0. We will give a preorder on the set of local isotopies of f such that

$$I \lesssim I' \text{ if and only if } I' \text{ is homotopic to } J^q I \text{ for a } q \geq 0.$$

Indeed, fix a sufficiently small r such that I and I' are well defined on D_r. Let $\widetilde{I'} = (\widetilde{f'}_t)_{t\in[0,1]}$ be the lift of $I'|_{D_r\setminus\{0\}}$ to $\mathbb{R} \times (-r, 0)$ such that $\widetilde{f}'_0$ is the identity. We write $I \lesssim I'$ if

$$p_1\widetilde{f}_1(\theta, y) \leq p_1\widetilde{f}'_1(\theta, y) \text{ for all } (\theta, y) \in \mathbb{R} \times (-r, 0),$$

where p_1 is the projection onto the first factor. Thus $\lesssim$ is a preorder, and

$$I \lesssim I' \text{ and } I' \lesssim I \Longleftrightarrow I \text{ is locally homotopic to } I'.$$

In this case, we will say that I and I' are *equivalent* and write $I \sim I'$.

More generally, we consider an orientation preserving local homeomorphism on an oriented surface. Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at a fixed point z_0 in a surface M. Let $h : (U, z_0) \to (U', 0)$ be a local homeomorphism. Then $h \circ I \circ h^{-1} = (h \circ f_t \circ h^{-1})_{t\in[0,1]}$ is a local isotopy at 0, and we define the *index of* I at z_0 to be $i(h \circ I \circ h^{-1}, 0)$, which is independent of the choice of h. We denote it by $i(I, z_0)$. Similarly, we have a preorder on the set of local isotopies of f at z_0.

The Lefschetz index at an isolated fixed point and the indices of the local isotopies are related. We have the following result:

Proposition 2.1. *([LC08][LR13]) Let $f : W \to W'$ be an orientation preserving homeomorphism with an isolated fixed point z. Then, we have the following results:*

- *if $i(f, z) \neq 1$, there exists a unique homotopy class of local isotopies such that $i(I, z) = i(f, z) - 1$ for every local isotopy I in this class, and the indices of the other local isotopies are equal to* 0*;*
- *if $i(f, z) = 1$, the indices of all the local isotopies are equal to* 0*.*

2.4 Brouwer plane translation theorem

In this section, we will recall the Brouwer plane translation theorem. More details can be found in [Bro12], [Gui94] and [Fra92].

Let f be an orientation preserving homeomorphism of $\mathbb{R}^2$. If f does not have any fixed point, the Brouwer plane translation theorem asserts that every $z \in \mathbb{R}^2$ is contained in a translation domain for f, i.e. an open connected set of $\mathbb{R}^2$ whose boundary is $L \cup f(L)$, where L is the image of a proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$ such that L separates $f(L)$ and $f^{-1}(L)$.

As an immediate corollary, one knows that if f is an orientation and area preserving homeomorphism of a plane [1] with finite area, it has at least one fixed point.

2.5 Transverse foliations and its index at an isolated end

In this section, we will introduce the index of a foliation at an isolated end. More details can be found in [LC08].

Let M be an oriented surface and $\mathcal{F}$ be an oriented topological foliation on M. For every point z, there is a neighborhood V of z and a homeomorphism $h : V \to (0,1)^2$ preserving the orientation such that the images of the leaves of $\mathcal{F}|_V$ are the vertical lines upward. We call V a *trivialization neighborhood* of z, and h a *trivialization chart*.

Let z_0 be an isolated end of M. We choose a small annulus $U \subset M$ such that z_0 is an end of U. Let $h : U \to \mathbb{D} \setminus \{0\}$ be a homeomorphism which sends z_0 to 0 and preserves the orientation. Let $\gamma : \mathbb{T}^1 \to \mathbb{D} \setminus \{0\}$ be a simple closed curve homotopic to $\partial\mathbb{D}$. We can cover the curve by finite trivialization neighborhoods $\{V_i\}_{1\leq i\leq n}$ of the foliation $\mathcal{F}_h$, where $\mathcal{F}_h$ is the image of $\mathcal{F}|_U$. For every $z \in V_i$, we denote by $\phi^+_{V_i,z}$ the positive half leaf of the the leaf in V_i containing z. Then we can construct a continuous map ψ from the curve γ

1. Here, a plane is an open set homeomorphic to $\mathbb{R}^2$.

to $\mathbb{D} \setminus \{0\}$, such that $\psi(z) \in \phi^+_{V_i,z}$ for all $0 \leq i \leq n$ and for all $z \in V_i$. We define the *index* $i(\mathcal{F}, z_0)$ of $\mathcal{F}$ at z_0 to be the Brouwer degree of the application

$$\theta \mapsto \frac{\psi(\gamma(\theta)) - \gamma(\theta)}{\|\psi(\gamma(\theta)) - \gamma(\theta)\|},$$

which depends neither on the choice of ψ, nor on the choice of V_i, nor on the choice of γ, nor on the choice of h.

We say that a path $\gamma : [0,1] \to M$ is *positively transverse* to $\mathcal{F}$, if for every $t_0 \in [0,1]$, there exists a trivialization neighborhood V of $\gamma(t_0)$ and $\varepsilon > 0$ such that $\gamma([t_0 - \varepsilon, t_0 + \varepsilon] \cap [0,1]) \subset V$ and $h \circ \gamma|_{[t_0-\varepsilon,t_0+\varepsilon]\cap[0,1]}$ intersects the vertical lines from left to right, where $h : V \to (0,1)^2$ is the trivialization chart.

Let f be a homeomorphism on M isotopic to the identity, and $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. We say that an oriented foliation $\mathcal{F}$ on M is a *transverse foliation* of I if for every $z \in M$, there is a path that is homotopic to the trajectory $t \to f_t(z)$ of z along I and is positively transverse to $\mathcal{F}$.

Suppose $I = (f_t)_{t\in[0,1]}$ is a local isotopy at z_0, we say that $\mathcal{F}$ is *locally transverse* to I if for every sufficiently small neighborhood U of z_0, there exists a neighborhood $V \subset U$ such that for all $z \in V \setminus \{z_0\}$, there exists a path in $U \setminus \{z_0\}$ that is homotopic to the trajectory $t \mapsto f_t(z)$ of z along I and is positively transverse to $\mathcal{F}$.

Proposition 2.2. *[LC08] Suppose that I is an identity isotopy on a surface M with an isolated end z and $\mathcal{F}$ is a transverse foliation of I. If M is not a plane, $\mathcal{F}$ is also locally transverse to the local isotopy I at z.*

Proposition 2.3. *[LC08] Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at an isolated fixed point z_0, I be a local isotopy of f at z_0, and $\mathcal{F}$ be a foliation that is locally transverse to I, then*

- *$i(\mathcal{F}, z_0) = i(I, z_0) + 1$;*
- *$i(f, z_0) = i(\mathcal{F}, z_0)$ if $i(\mathcal{F}, z_0) \neq 1$.*

2.6 The existence of a transverse foliation and Jaulent's preorder

Let f be a homeomorphism of M isotopic to the identity, and $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. A *contractible fixed point* z of f associated to I is a fixed point of f such that the trajectory of z along I, that is the path $t \mapsto f_t(z)$, is a loop homotopic to zero in M.

Theorem 2.4. *[LC05] If $I = (f_t)_{t\in[0,1]}$ is an identity isotopy of a homeomorphism f of M such that there exists no contractible fixed point of f associated to I, then there exists a transverse foliation $\mathcal{F}$ of I.*

One can extend this result to the case where there exist contractible fixed points by defining the following preorder of Jaulent [Jau14].

Let us denote by $\mathrm{Fix}(f)$ the set of fixed points of f, and for every identity isotopy $I = (f_t)_{t\in[0,1]}$ of f, by $\mathrm{Fix}(I) = \cap_{t\in[0,1]}\mathrm{Fix}(f_t)$ the set of fixed points of I. Let X be a closed subset of $\mathrm{Fix}(f)$. We denote by (X, I_X) the couple that consists of a closed subset $X \subset \mathrm{Fix}(f)$ such that $f|_{M\setminus X}$ is isotopic to the identity and an identity isotopy I_X of $f|_{M\setminus X}$.

Let $\pi_X : \widetilde{M}_X \to M \setminus X$ be the universal cover, and $\widetilde{I}_X = (\widetilde{f}_t)_{t\in[0,1]}$ be the identity isotopy that lifts I_X. We say that $\widetilde{f}_X = \widetilde{f}_1$ is *the lift of f associated to I_X*. We say that a path $\gamma : [0,1] \to M \setminus X$ from z to $f(z)$ is *associated* to I_X if there exists a path $\widetilde{\gamma} : [0,1] \to \widetilde{M}_X$ that is the lift of γ and satisfies $\widetilde{f}_X(\widetilde{\gamma}(0)) = \widetilde{\gamma}(1)$. We write $(X, I_X) \precsim (Y, I_Y)$, if

- $X \subset Y \subset (X \cup \pi_X(\mathrm{Fix}(\widetilde{f}_X)))$;
- all the paths in $M \setminus Y$ associated to I_Y are also associated to I_X.

The preorder $\precsim$ is well defined. Moreover, if one has $(X, I_X) \precsim (Y, I_Y)$ and $(Y, I_Y) \precsim (X, I_X)$, then one knows $X = Y$ and $\widetilde{f}_X = \widetilde{f}_Y$. In this case, we will write $(X, I_X) \sim (Y, I_Y)$. Jaulent proved that two couples (X, I_X) and (X, I'_X) are always equivalent if $M \setminus X$ is not homeomorphic to an annulus or a torus.

When the closed subset $X \subset \mathrm{Fix}(f)$ is totally disconnected, an identity isotopy I_X on $M \setminus X$ can be extended to an identity isotopy on M that fixes every point in X; but when X is not totally disconnected, one may fail to find such an extension. A necessary condition for the existence of such an extension is that for every closed subset $Y \subset X$, there exists (Y, I_Y) that satisfies $(Y, I_Y) \precsim (X, I_X)$. By a result (unpublished yet) due to Béguin, Crovisier and Le Roux, this condition is also sufficient to prove the existence of an identity isotopy I' of f on M that fixes every point in X and satisfies $(X, I_X) \sim (X, I'|_{M\setminus X})$ (here, we do not know whether I_X can be extended). Formally, we denote by $\mathcal{I}$ the set of couples (X, I_X) such that for all closed subset $Y \subset X$, there exists (Y, I_Y) that satisfies $(Y, I_Y) \precsim (X, I_X)$. Then, we have the following results:

Proposition 2.5. *[BCLR][2] For $(X, I_X) \in \mathcal{I}$, there exists an identity isotopy I' of f on M that fixes every point in X and satisfies $(X, I_X) \sim (X, I'|_{M\setminus X})$.*

Proposition 2.6. *[Jau14] Let $(X, I_X) \in \mathcal{I}$, and $\widetilde{f}_X$ be the lift of $f|_{M\setminus X}$ to $\widetilde{M}_X$ associated to I_X. If $z \in \mathrm{Fix}(f) \setminus \mathrm{Fix}(I)$ is a fixed point of f such that $\widetilde{f}_X$ fixes all the points in $\pi_X^{-1}\{z\}$, then there exists $(X \cup \{z\}, I_{X\cup\{z\}}) \in \mathcal{I}$ such that $(X, I_X) \precsim (X \cup \{z\}, I_{X\cup\{z\}})$. In particular, if (X, I_X) is maximal in $(\mathcal{I}, \precsim)$, $f|_{M\setminus X}$ has no contractible fixed point associated to I_X.*

Proposition 2.7. *[Jau14] If $\{(X_\alpha, I_{X_\alpha})\}_{\alpha\in J}$ is a totally ordered chain in $(\mathcal{I}, \precsim)$, then there exists $(X_\infty, I_{X_\infty}) \in \mathcal{I}$ that is an upper bound of the this chain, where $X_\infty = \overline{\cup_{\alpha\in J} X_\alpha}$*

Theorem 2.8. *[Jau14] If I is an identity isotopy of a homeomorphism f on M, then there exists a maximal $(X, I_X) \in \mathcal{I}$ such that $(\mathrm{Fix}(I), I) \precsim (X, I_X)$. Moreover, $f|_{M\setminus X}$ has no contractible fixed point associated to I_X, and there exists a transverse foliation $\mathcal{F}$ of I_X on $M \setminus X$.*

Remark 2.9. Here, we can also consider the previous foliation $\mathcal{F}$ to be a singular foliation on M whose singularities are the points in X. Moreover, $\mathcal{F}$ is also a transverse foliation of I'_X for all $(X, I'_X) \sim (X, I_X)$. In particular, if I_X is the restriction to $M \setminus X$ of an identity isotopy I' on M, we will say that $\mathcal{F}$ a transverse (singular) foliation of I'.

Remark 2.10. In this article, we denote also by I_X an identity isotopy on M that fixes all the points in X, when there is no ambiguity. Proposition 2.6, 2.7 and Theorem 2.8 are still valid if we replace the definition of $\mathcal{I}$ with the set of couples (X, I_X) of a closed subset $X \subset \mathrm{Fix}(f)$ and an identity isotopy I_X on M that fixes every point in X. When $\mathrm{Fix}(f)$ is totally disconnected, it is obvious; when $\mathrm{Fix}(f)$ is not totally disconnected, we should admit Proposition 2.5.

2. It is a talk of Crovisier in the conference "Surfaces in Sao Paulo" in April, 2014.

We call $(Y, I_Y) \in \mathcal{I}$ an *extension* of (X, I_X) if $(X, I_X) \precsim (Y, I_Y)$; we call I' an *extension* of $(X, I_X) \in \mathcal{I}$ if $(X, I_X) \precsim (\mathrm{Fix}(I'), I')$; we call I' an *extension* of I if I' is an extension of $(\mathrm{Fix}(I), I)$. We say that I' is a *maximal extension* if $(\mathrm{Fix}(I'), I')$ is maximal in $(\mathcal{I}, \precsim)$.

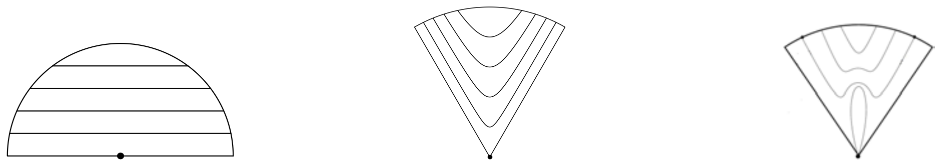
In particular, when M is a plane, Béguin, Crovisier and Le Roux proved the following result (unpublished yet).

Proposition 2.11. *[BCLR] If f is an orientation preserving homeomorphism of the plane, and if $X \subset \mathrm{Fix}(f)$ is a connected and closed subset, then there exists an identity isotopy I of f such that $X \subset \mathrm{Fix}(I)$.*

2.7 Dynamics of an oriented foliation in a neighborhood of an isolated singularity

In this section, we consider singular foliations. A *sink* (resp. a*source*) of $\mathcal{F}$ is an isolated singular point of $\mathcal{F}$ such that there is a homeomorphism $h : U \to \mathbb{D}$ which sends z_0 to 0 and sends the restricted foliation $\mathcal{F}|_{U \setminus \{z_0\}}$ to the radial foliation of $\mathbb{D} \setminus \{0\}$ with the leaves toward (resp. backward) 0, where U is a neighborhood of z_0 and $\mathbb{D}$ is the unit disk. A *petal* of $\mathcal{F}$ is a closed topological disk whose boundary is the union of a leaf and a singularity. Let $\mathcal{F}_0$ be the foliation on $\mathbb{R}^2 \setminus \{0\}$ whose leaves are the horizontal lines except the $x-$axis which is cut into two leaves. Let $S_0 = \{y \geq 0 : x^2 + y^2 \leq 1\}$ be the half-disk. We call a closed topological disk S a *hyperbolic sector* if there exist

- a closed set $K \subset S$ such that $K \cap \partial S$ is reduced to a singularity z_0 and $K \setminus \{z_0\}$ is the union of the leaves of $\mathcal{F}$ that are contained in S,
- and a continuous map $\phi : S \to S_0$ that maps K to 0 and the leaves of $\mathcal{F}|_{S \setminus K}$ to the leaves of $\mathcal{F}_0|_{S_0}$.



(a) the hyperbolic sector model S_0 (b) a pure hyperbolic sector (c) a strange hyperbolic sector

Figure 2.1: The hyperbolic sectors

Le Roux gives a description of the dynamics of an oriented foliation $\mathcal{F}$ near an isolated singularity z_0.

Proposition 2.12. *[LR13] We have one of the following cases:*

i) (sink or source) there exists a neighborhood of z_0 that contains neither a closed leaf, nor a petal, nor a hyperbolic sector;

ii) (cycle) every neighborhood of z_0 contains a closed leaf;

iii) (petal) every neighborhood of z_0 contains a petal, and does not contain any hyperbolic sector;

iv) (saddle) every neighborhood of z_0 contains a hyperbolic sector, and does not contain any petal;

v) (mixed) every neighborhood of z_0 contains both a petal and a hyperbolic sector.

Moreover, $i(\mathcal{F}, z_0)$ is equal to 1 in the first two cases, is strictly bigger than 1 in the petal case, and is strictly smaller than 1 in the saddle case.

Remark 2.13. In particular, let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at z_0, I be a local isotopy of f, $\mathcal{F}$ be an oriented foliation that is locally transverse to I, and z_0 be an isolated singularity of $\mathcal{F}$. If P is a petal in a small neighborhood of z_0 and ϕ is the leaf in ∂P, then $\phi \cup \{z_0\}$ divides M into two parts. We denote by $L(\phi)$ the one to the left and $R(\phi)$ the one to the right. By definition, P contains the positive orbit of $R(\phi) \cap L(f(\phi))$ or the negative orbit of $L(\phi) \cap R(f^{-1}(\phi))$. Then, a petal in a small neighborhood of z_0 contains the positive or the negative orbit of a wandering open set. So does the topological disk whose boundary is a closed leaf in a small neighborhood of z_0. So, if f is area preserving, or if there exists a neighborhood of z_0 that contains neither the positive nor the negative orbit of any wandering open set, then z_0 is either a sink, a source, or a saddle of $\mathcal{F}$.

In some particular cases, the local dynamics of a transverse foliation can be easily deduced. We have the following result:

Proposition 2.14. *[LC08] Let I be a local isotopy at z_0 such that $i(I, z_0) \neq 0$. If I' is another local isotopy at z_0 and if $\mathcal{F}'$ is an oriented foliation that is locally transverse to I'. Then,*

- *the indices $i(I', z_0)$ and $i(I, z_0)$ are equal if $I' \sim I$;*
- *z_0 is a sink of $\mathcal{F}'$ if $I' > I$;*
- *z_0 is a source of $\mathcal{F}'$ if $I' < I$.*

2.8 The local rotation type of a local isotopy

In this section, suppose that $f : (W, z_0) \to (W', z_0)$ is an orientation and area preserving local homeomorphism at an isolated fixed point z_0, and that I is a local isotopy of f. We say that I has a *positive rotation type* (resp. *negative rotation type*) if there exists a locally transverse foliation $\mathcal{F}$ of I such that z_0 is a sink (resp. source) of $\mathcal{F}$. Matsumoto [Mat01] proved the following result:

Proposition 2.15. *[Mat01] If $i(f, z_0)$ is equal to 1, I has only one of the two kinds of rotation types.*

Remark 2.16. By considering the index of foliation, one deduces the following corollary: *if $\mathcal{F}$ and $\mathcal{F}'$ are two locally transverse foliations of I, and if 0 is a sink (resp. source) of $\mathcal{F}$, then 0 is a sink (resp. a source) of $\mathcal{F}'$.*

2.9 Prime-ends compactification and rotation number

In this section, we first recall some facts and definitions from Carathéodory's prime-ends theory, and then give the definition of the prime-ends rotation number. More details can be found in [Mil06] and [KLCN14].

Let $U \subsetneq \mathbb{R}^2$ be a simply connected domain, then there exists a natural compactification of U by adding a circle, that can be defined in different ways. One explanation is the following: we can identify $\mathbb{R}^2$ with $\mathbb{C}$ and consider a conformal diffeomorphism

$$h : U \to \mathbb{D},$$

where $\mathbb{D}$ is the unit disk. We endow $U \sqcup S^1$ with the topology of the pre-image of the natural topology of $\overline{\mathbb{D}}$ by the application

$$\overline{h} : U \sqcup S^1 \to \overline{\mathbb{D}},$$

whose restriction is h on U and the identity on S^1 .

Any arc in U which lands at a point z of ∂U corresponds, under h, to an arc in $\mathbb{D}$ which lands at a point of S^1, and arcs which land at distinct points of ∂U necessarily correspond to arcs which land at distinct points of S^1. We define an *end-cut* to be the image of a simple arc $\gamma : [0, 1) \to U$ with a limit point in ∂U. Its image by h has a limit point in S^1. We say that two end-cuts are *equivalent* if their images have the same limit point in S^1. We say that a point $z \in \partial U$ is *accessible* if there is an end-cut that lands at z. Then the set of points of S^1 that are limit points of an end-cut is dense in S^1, and accessible points of ∂U are dense in ∂U. We define a *cross-cut* by the image of a simple arc $\gamma : (0, 1) \to U$ which extends to an arc $\overline{\gamma} : [0, 1] \to \overline{U}$ joining two points of ∂U and such that each of the two components of $U \setminus \gamma$ has a boundary point in $\partial U \setminus \overline{\gamma}$.

Let f be an orientation preserving homeomorphism of U. We can extend f to a homeomorphism of the prime-ends compactification $U \sqcup S^1$, and denote it by $\overline{f}$. In fact, for a point $z \in S^1$ which is a limit point of an end-cut γ, we can naturally define $\overline{f}(z)$ to be the limit point of $f \circ \gamma$. Then we can define the *prime-ends rotation number* $\rho(f, U) \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ to be the Poincaré's rotation number of $\overline{f}|_{S^1}$. In particular, if f fixes every point in ∂U, $\rho(f, U) = 0$.

Let $K \subset \mathbb{R}^2$ be a continuum, and U_K be the unbounded component of $\mathbb{R}^2 \setminus K$. Then, U_K is an annulus and becomes a simply connected domain of the Riemann sphere if we identify $\mathbb{R}^2$ with $\mathbb{C}$ and add a point at infinity. The prime-ends compactification also gives us a compactification of the end of U_K corresponding to K by adding the circle of prime-ends. We can define *end-cuts* and *cross-cuts* similarly.

Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at $0 \in \mathbb{R}^2$, and $K \subset W$ be an invariant continuum containing 0. Similarly, we can naturally extend $f|_{U_K \cap W}$ to a homeomorphism $f_K : U_K \cap W \cup S^1 \to U_K \cap W' \cup S^1$, and define the *rotation number* $\rho(f, K) \in \mathbb{R}/\mathbb{Z}$ to be the Poincaré's rotation number of $f_K|_{S^1}$.

Furthermore, if $I = (f_t)_{t\in[0,1]}$ is a local isotopy of f at 0, we consider the universal covering projections

$$\begin{aligned} \pi : \mathbb{R} \times (-\infty, 0) &\to \mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\} \\ (\theta, y) &\mapsto -ye^{i2\pi\theta} \end{aligned}$$

and

$$\begin{aligned} \pi' : \mathbb{R} &\to S^1 \\ \theta &\mapsto e^{i2\pi\theta}. \end{aligned}$$

Let $\widetilde{U}_K = \pi^{-1}(U_K)$, $\widetilde{W} = \pi^{-1}(W)$, and $\widetilde{W}' = \pi^{-1}(W')$. Let

$$\pi_K : \widetilde{U}_K \sqcup \mathbb{R} \to U_K \sqcup \mathbb{T}^1$$

be the map such that $\pi_K = \pi$ in $\widetilde{U}_K$ and $\pi_K = \pi'$ on $\mathbb{R}$. We endow the topology on $\widetilde{U}_K \sqcup R$ such that π_K is a universal cover. Let $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ be the lift of $(f_t|_{V\setminus\{0\}})_{t\in[0,1]}$ such that $\widetilde{f}_0$ is the identity, where V is a small neighborhood of 0. Let $\widetilde{f} : \widetilde{W} \to \widetilde{W}'$ be the lift of $f|_{W\setminus\{0\}}$ such that $\widetilde{f} = \widetilde{f}_1$ in $\pi^{-1}(V)$, we call it the *lift of f associated to I*. Let $\widetilde{f}_K : (\widetilde{W} \cap \widetilde{U}_K) \sqcup \mathbb{R} \to (\widetilde{W}' \cap \widetilde{U}_K) \sqcup \mathbb{R}$ be the lift of f_K such that $\widetilde{f}_K = \widetilde{f}$ in $\widetilde{W} \cap \widetilde{U}_K$, we call it the *lift of f_K associated to I*. We define the *rotation number* $\rho(I, K) = \lim_{n\to\infty} \dfrac{\widetilde{f}_K^n(\theta) - \theta}{n}$ which is a real number that does not depend on the choice of θ. We know that $\rho(I, K)$ is a representative of $\rho(f, K)$ in $\mathbb{R}$.

We have the following property:

Proposition 2.17. *[LC03] Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at a non-accumulated indifferent point 0. Let $U \subset \overline{U} \subset W$ be a Jordan domain such that $\overline{U}$ does not contain any periodic orbit except 0, and that for all $V \subset U$, the connected component of $\cap_{n\in\mathbb{Z}} f^{-n}(\overline{V})$ containing 0 intersects the boundary of V. Let K_0 be the connected component of $\cap_{n\in\mathbb{Z}} f^{-n}(\overline{U})$ containing 0. Then for every local isotopy I of f, and for every invariant continuum $K \subset \overline{U}$ containing 0, one has $\rho(I, K) = \rho(I, K_0)$.*

This proposition implies that if $f : (W, 0) \to (W', 0)$ is an orientation preserving local homeomorphism at a non-accumulated indifferent point 0, we can define the *rotation number* $\rho(I, 0)$ for every local isotopy I of f at 0, by writing $\rho(I, 0) = \rho(I, K)$ where K is a non-trivial invariant continuum sufficiently close to 0.

More generally, if $f : (W, z_0) \to (W', z_0)$ is an orientation preserving local homeomorphism at a non-accumulated indifferent point $z_0 \in M$, we can conjugate it to a local homeomorphism at 0, and get the previous definitions and results similarly.

2.10 The local rotation set

In this section, we will give a definition of the local rotation set and will describe the relations between the rotation set and the rotation number. More details can be found in [LR13].

Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at $0 \in \mathbb{R}^2$, and $I = (f_t)_{t\in[0,1]}$ be a local isotopy of f. Given two neighborhoods $V \subset U$ of 0 and an integer $n \geq 1$, we define

$$E(U, V, n) = \{z \in U : z \notin V, f^n(z) \notin V, f^i(z) \in U \text{ for all } 1 \leq i \leq n\}.$$

We define the *rotation set* of I relative to U and V by

$$\rho_{U,V}(I) = \cap_{m\geq 1}\overline{\cup_{n\geq m}\{\rho_n(z), z \in E(U, V, n)\}} \subset [-\infty, \infty],$$

where $\rho_n(z)$ is the average change of angular coordinate along the trajectory of z. More precisely, let

$$\begin{array}{rcl} \pi : \mathbb{R} \times (-\infty, 0) & \to & \mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\} \\ (\theta, y) & \mapsto & -ye^{i2\pi\theta} \end{array}$$

be the universal covering projection, $\widetilde{f} : \pi^{-1}(W) \to \pi^{-1}(W')$ be the lift of f associated to I, and $p_1 : \mathbb{R} \times (-\infty, 0) \to \mathbb{R}$ be the projection onto the first factor. We define

$$\rho_n(z) = \frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n},$$

where $\widetilde{z}$ is any point in $\pi^{-1}\{z\}$.

We define the *local rotation set* of I to be

$$\rho_s(I, 0) = \cap_U \overline{\cup_V \rho_{U,V}(I)} \subset [-\infty, \infty],$$

where $V \subset U \subset W$ are neighborhoods of 0.

We say that f can be *blown-up* at 0 if there exists an orientation preserving homeomorphism $\Phi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{T}^1 \times (-\infty, 0)$, such that $\Phi f \Phi^{-1}$ can be extended continuously to $\mathbb{T}^1 \times \{0\}$. We denote this extension by h. Suppose that f is not conjugate the contraction $z \mapsto \frac{z}{2}$ or the expansion $z \mapsto 2z$. We define the *blow-up rotation number* $\rho(f, 0)$ of f at

0 to be the Poincaré rotation number of $h|_{\mathbb{T}^1}$. Let $I=(f_t)_{t\in[0,1]}$ be a local isotopy of f, $(\widetilde{h}_t)$ be the natural lift of $\Phi|_{\mathbb{T}^1\times(0,r)}\circ f_t|_{D_r\setminus\{0\}}\circ\Phi^{-1}|_{\mathbb{T}^1\times(0,r)}$, where D_r is a sufficiently small disk with radius r and centered at 0, and $\widetilde{h}$ be the lift of h such that $\widetilde{h}=\widetilde{h}_1$ in a neighborhood of $\mathbb{R}\times\{0\}$. We define the *blow-up rotation number* $\rho(I,0)$ *of* I *at* 0 to be the rotation number of $h|_{\mathbb{T}^1}$ associated to the lift $\widetilde{h}|_{\mathbb{R}\times\{0\}}$, which is a representative of $\rho(f,0)$ on $\mathbb{R}$. J-M Gambaudo, Le Calvez and E. Pécou [GLCP96] proved that neither $\rho(f,0)$ nor $\rho(I,0)$ depend on the choice of Φ, which generalizes a previous result of Naĭshul′ [Naĭ82]. In particular, if f is a diffeomorphism, f can be blown-up at 0 and the extension of f on $\mathbb{T}^1$ is induced by the map

$$v\mapsto\frac{Df(0)v}{\|Df(0)v\|}$$

on the space of unit tangent vectors.

More generally, if $f:(W,z_0)\to(W',z_0)$ is an orientation preserving local homeomorphism at z_0 that is not conjugate to the contraction or the expansion, we can give the previous definitions for f by conjugate it to an orientation preserving local homeomorphism at $0\in\mathbb{R}^2$.

The local rotation set can be empty. However, due to Le Roux [LR08], we know that the rotation set is not empty if f is area preserving. More precisely, we have the following result:

Proposition 2.18. *[LR13] Let $f:(W,z_0)\to(W',z_0)$ be an orientation preserving local homeomorphism at z_0, and $I=(f_t)_{t\in[0,1]}$ be a local isotopy of f. Then $\rho_s(I,z_0)$ is empty if and only if f is conjugate to one of the following maps*

- *the contraction $z\mapsto\frac{z}{2}$;*
- *the expansion $z\mapsto 2z$;*
- *a holomorphic function $z\mapsto e^{i2\pi\frac{p}{q}}z(1+z^{qr})$ with $q,r\in\mathbb{N}$ and $p\in\mathbb{Z}$.*

Remark 2.19. In the three cases, f can be blown-up at z_0. But $\rho(f,z_0)$ is defined only in the third case. More precisely, $\rho(f,z_0)$ is equal to $\frac{p}{q}+\mathbb{Z}$. Moreover, if I is conjugate to $z\mapsto z^{i2\pi t\frac{p}{q}}(1+tz^{qr})$, then $\rho(I,z_0)$ is equal to $\frac{p}{q}$.

We say that z is a *contractible* fixed point of f associated to the local isotopy $I=(f_t)_{t\in[0,1]}$ if the trajectory $t\mapsto f_t(z)$ of z along I is a loop homotopic to zero in $W\setminus\{z_0\}$. We say that f satisfies the *local intersection condition*, if there exists a neighborhood of z_0 that does not contain any simple closed curve which is the boundary of a Jordan domain containing z_0 and does not intersect its image by f. In particular, if f is area preserving or if there exists a neighborhood of z_0 that contains neither the positive nor the negative orbit of any wandering open set, f satisfies the local intersection condition.

The local rotation set satisfies the following properties:

Proposition 2.20. *[LR13] Let $f:(W,z_0)\to(W',z_0)$ be an orientation preserving local homeomorphism at z_0, and I be a local isotopy of f at z_0. One has the following results:*

i) for all integer p,q, $\rho_s(J_{z_0}^pI^q,z_0)=q\rho_s(I,z_0)+p$;

ii) if z_0 is accumulated by contractible fixed points of f associated to I, then $0\in\rho_s(I,z_0)$;

iii) if f satisfies the local intersection condition and if 0 is an interior point of the convex hull of $\rho_s(I,z_0)$, then z_0 is accumulated by contractible fixed points of f associated to I;

iv) if I has a positive (resp. negative) rotation type, then $\rho_s(I,z_0)\subset[0,+\infty]$ (resp. $\rho_s(I,z_0)\subset[-\infty,0]$);

v) if $\rho_s(I, z_0)$ is a non-empty set that is contained in $(0, +\infty]$ (resp. $[-\infty, 0)$), then I has a positive (resp.negative) rotation type;

vi) if $\rho_s(I, z_0)$ is a non-empty set that is contained in $[0, \infty]$ (resp. $[-\infty, 0]$) and is not reduced to 0, and if z_0 is not accumulated by contractible fixed points of f associated to I, then I has a positive (resp. negative) rotation type;

vii) if f can be blown-up at z_0, and if $\rho_s(I, z_0)$ is not empty, then $\rho_s(I, z_0)$ is reduced to the single real number $\rho(I, z_0)$.

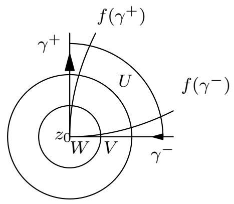
viii) if z_0 is a non-accumulated indifferent point, $\rho_s(I, z_0)$ is reduced to $\rho(I, z_0)$ (the rotation number defined in Section 2.9).

Remark 2.21. When f satisfies the local intersection condition, one can deduce that $\rho_s(I, z_0)$ is a closed interval as a corollary of the assertion i), ii), iii) of the proposition. In general case, due to an unpublished work of Jonathan Conejeros, $\rho_s(I, z_0)$ is always a closed interval.

Remark 2.22. Le Roux also gives several criteria implying that f can be blown-up at z_0. The one we need in this article is due to Béguin, Crovisier and Le Roux [LR13]:

If there exists an arc γ at z_0 whose germ is disjoint with the germs of $f^n(\gamma)$ for all $n \neq 0$, then f can be blown-up at z_0.

In particular, if there exists a leaf γ^+ from z_0 and a leaf γ^- toward z_0 (we are in this case if z_0 is a petal, a saddle, or a mixed singularity of $\mathcal{F}$), we can choose a sector U as in the picture. Let V be a small neighborhood of z_0. There exists a neighborhood $W \subset V$



of z_0 such that

$$f(\overline{U} \cap W) \subset (\mathrm{Int}(U) \cap V) \cup \{z_0\}.$$

So, the germs at z_0 of $f^n(\gamma^+)$ are pairwise disjoint, and hence f can be blown-up at z_0. Moreover, $\rho(I, z_0)$ is equal to 0 in this case.

Le Roux also studied the dynamics near a non-accumulated saddle point, and proved the following result:

Proposition 2.23. *[LR13] If z_0 is a non-accumulated saddle point, then f can be blown-up at z_0 and $\rho_s(I, z_0)$ is reduced to a rational number. Moreover, if $i(f, z_0)$ is equal to 1, this rational number is not an integer.*

2.11 Some generalizations of Poincaré-Birkhoff theorem

In this section, we will introduce several generalizations of Poincaré-Birkhoff theorem. An *essential* loop in the annulus is a loop that is not homotopic to zero.

We first consider the homeomorphisms of closed annuli. Let f be a homeomorphism of $\mathbb{T}^1 \times [0, 1]$ isotopic to the identity, $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. Let $\pi : \mathbb{R} \times [0, 1] \to \mathbb{T}^1 \times \mathbb{R}$ be the universal cover, $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ be the identity isotopy that lifts I, $\widetilde{f} = \widetilde{f}_1$ be the lift of f associated to I, and $p_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first factor. The limits

$$\lim_{n\to\infty} \frac{p_1 \circ \widetilde{f}^n(x,0) - x}{n} \quad \text{and} \quad \lim_{n\to\infty} \frac{p_1 \circ \widetilde{f}^n(x,1) - x}{n}$$

exists for all $x \in \mathbb{R}$, and do not depend on the choice of x. We define the *rotation number* of f on each boundary to be the corresponding limits. We define the *rotation number* of a point $z \in \mathbb{T}^1 \times [0,1]$ associated to I to be the limit

$$\lim_{n\to+\infty} \frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n},$$

if this limit exists. We say that f satisfies the *intersection property* if $f \circ \gamma$ intersects γ, for every simple essential loop $\gamma \subset \mathbb{T}^1 \times (0,1)$. We have the following generalizations of Poincaré-Birkhoff theorem:

Proposition 2.24. *[Bir26] Let f be a homeomorphism of $\mathbb{T}^1 \times [0,1]$ isotopic to the identity and satisfying the intersection property. If the rotation number of f on the two boundaries are different, then there exists a q-periodic orbit of rotation number p/q for all irreducible rational $p/q \in (\rho_1, \rho_2)$, where ρ_1 and ρ_2 are the rotation numbers of f on the boundaries.*

We also consider homeomorphisms of open annuli. Let $f : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ be a homeomorphism isotopic to the identity, and $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. Let $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ be the universal cover, $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ be the identity isotopy that lifts I, $\widetilde{f} = \widetilde{f}_1$ be the lift of f associated to I, and $p_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first factor. Similarly, we define the *rotation number* of a recurrent point $z \in \mathbb{T}^1 \times \mathbb{R}$ associated to I to be the limit

$$\lim_{n\to+\infty} \frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n} \in [-\infty, \infty],$$

if this limit exists. We say that f satisfies the *intersection property* if $f \circ \gamma$ intersects γ, for every simple essential loop $\gamma \subset \mathbb{T}^1 \times \mathbb{R}$. Then, we have the following generalization of Poincaré-Birkhoff theorem:

Proposition 2.25 ([Fra88], [LC05])**.** *Let $f : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ be a homeomorphism isotopic to the identity and satisfying the intersection property. If there exist two recurrent points of rotation numbers $\rho_1, \rho_2 \in [-\infty, +\infty]$ respectively such that $\rho_1 < \rho_2$, then there exists a q-periodic orbit of rotation number p/q for all irreducible rational $p/q \in (\rho_1, \rho_2)$.*

Remark 2.26. The result is also true for area preserving homeomorphisms of the closed or half closed annulus by considering a symmetry.

We will also need the following generalization due to Lucien Guillou, in which case the boundary of the annulus is not fixed:

Proposition 2.27. *[Gui94] Let $f : \mathbb{T}^1 \times [-a, a] \to \mathbb{T}^1 \times [-b, b]$, where $0 < a < b$, be an embedding homotopic to the inclusion, and $\widetilde{f} : \mathbb{R} \times [-a, a] \to \mathbb{R} \times [-b, b]$ be the lift of f. If f does not have any fixed point, and if $\widetilde{f}$ satisfies*

$$(p_1(\widetilde{f}(x,a)) - x)(p_1(\widetilde{f}(x', -a)) - x') < 0, \ \textit{for all } x, x' \in \mathbb{R},$$

then there exists an essential loop γ in $\mathbb{T}^1 \times [-a, a]$ such that $f(\gamma) \cap \gamma = \emptyset$.

2.12 Topologically monotone periodic orbits for annulus homeomorphisms

In this section, we will recall the braid type of a periodic orbit and the existence of the topologically monotone periodic orbits for annulus homeomorphisms under some conditions. More details can be found in [Boy92].

Denote by $\mathbb{A}$ the closed annulus $\mathbb{T}^1 \times [0,1]$. Let f be a homeomorphism of $\mathbb{A}$ that preserves the orientation and each boundary circle, and $\widetilde{f}$ be a lift of f to the universal cover $\widetilde{\mathbb{A}} = \mathbb{R} \times [0,1]$. Given $\widetilde{z} \in \widetilde{\mathbb{A}}$, we define its rotation number under $\widetilde{f}$ as

$$\rho(\widetilde{z}, \widetilde{f}) = \lim_{n\to\infty} \frac{p_1(\widetilde{f}^n(\widetilde{z})) - p_1(\widetilde{z})}{n},$$

if this limit exists, where p_1 is the projection onto the first factor. We define the rotation set of $\widetilde{f}$ to be

$$\rho(\widetilde{f}) = \{\rho(\widetilde{z}, \widetilde{f}), \widetilde{z} \in \widetilde{A}\}.$$

In particular, if I is an identity isotopy of f and $\widetilde{f}$ is the lift of f associated to I, this definition of the rotation number coincides with the definition of the rotation number in Section 2.11.

Fix a copy of the closed annulus minus n interior points, and denote it by $\mathbb{A}_n$. Let G_n be the group of isotopy classes of orientation preserving homeomorphism of $\mathbb{A}_n$. If O is an n-periodic orbit of f in the interior of $\mathbb{A}$, then there is an orientation preserving homeomorphism $h : \mathbb{A} \setminus O \to \mathbb{A}_n$. Philip Boyland defined the *braid type* $bt(O, f)$ to be the conjugacy class of $[h \circ f|_{\mathbb{A}\setminus O} \circ h^{-1}]$ in G_n, this isotopy class is independent of the choice of h. If O is an n-periodic orbit of f contained in a boundary circle of $\mathbb{A}$, he extends f near this boundary and gets a homeomorphism $\overline{f}$ also on a closed annulus. Then O is in the interior of this new annulus. The braid type $bt(O, \overline{f})$ is independent of the choice of the extension, and Boyland defined $bt(O, f) = bt(O, \overline{f})$.

Let p/q be an irreducible positive rational, and $\widetilde{T}_{p/q}$ be the homeomorphism of $\widetilde{\mathbb{A}}$ defined by $(x, y) \mapsto (x + p/q, y)$. It descends to a homeomorphism $T_{p/q}$ of $\mathbb{A}$. We denote by $\alpha_{p/q}$ the braid type $bt(O, T_{p/q})$, where O is any periodic orbit of $T_{p/q}$. We say that a q-periodic orbit O of f is a (p, q)-periodic orbit if $\rho(\widetilde{z}, \widetilde{f}) = p/q$ for any $\widetilde{z}$ in the lift of O. We say that a (p, q)-periodic orbit O is *topologically monotone* if $bt(O, f) = \alpha_{p/q}$. We define the *Farey interval* $I(p/q)$ of p/q to be the closed interval

$$[\max\{r/s : r/s < p/q, 0 < s < q, \text{ and } (r, s) = 1\}, \min\{r/s : r/s > p/q, 0 < s < q, \text{ and } (r, s) = 1\}].$$

In particular, the Farey interval of $1/q$ is equal to $[0, 1/(q-1)]$.

Boyland proved the following result:

Proposition 2.28 ([Boy92])**.** *If f is an orientation and boundary preserving homeomorphism of the closed annulus, and $p/q \in \rho(\widetilde{f})$ is an irreducible positive rational, then f has a (p, q)-topologically monotone periodic orbit. If f has a (p, q)-orbit that is not topologically monotone, then $I(p/q) \subset \rho(\widetilde{f})$.*

2.13 Annulus covering projection

Let M be an oriented surface, $X_0 \subset M$ be a closed set, and $z_0 \in M \setminus X_0$. Denote by M_0 the connected component of $M \setminus X_0$ containing z_0. Let $V \subset U \subset M_0$ be two small Jordan domains containing z_0. Write $\dot{U} = U \setminus \{z_0\}$ and $\dot{V} = V \setminus \{z_0\}$. Fix $z_1 \in \dot{V}$.

Let $\gamma \subset \dot{V}$ be a simple loop at z_1 such that the homotopic class $[\gamma]$ of γ in $\dot{V}$ generates $\pi_1(\dot{V}, z_1)$. Let $i : \dot{U} \to M_0 \setminus \{z_0\}$ be the inclusion, then $i_*\pi_1(\dot{U}, z_1)$ is a subgroup of $\pi_1(M_0\setminus\{z_0\}, z_1)$. Then, there exists a covering projection $\pi : (\widetilde{M}, \widetilde{z}_1) \to (M_0\setminus\{z_0\}, z_1)$ such that $\pi_*\pi_1(\widetilde{M}, \widetilde{z}_1) = i_*\pi_1(\dot{U}, z_1)$ by Theorem 2.13 in [Spa66]. Moreover, the fundamental group of $\widetilde{M}$ is isomorphic to $\mathbb{Z}$, so $\widetilde{M}$ is an annulus.

Let $\widetilde{\dot{U}}$ be the component of $\pi^{-1}(\dot{U})$ containing $\widetilde{z}_1$. Then $\pi_*\pi_1(\widetilde{\dot{U}}, \widetilde{z}_1) = \pi_1(\dot{U}, z_1)$ and the restriction of π to $\widetilde{\dot{U}}$ is a homeomorphism between $\widetilde{\dot{U}}$ and $\dot{U}$ by Theorem 2.9 in [Spa66]. Consider the ideal-ends compactification of $\widetilde{M}$, and denote by $\star$ the end in $\widetilde{\dot{U}}$. Then $\pi|_{\widetilde{\dot{U}}}$ can be extended continuously to a homeomorphism between $\widetilde{\dot{U}} \cup \{\star\}$ and U. We denote it by h.

If f is an orientation preserving homeomorphism of M_0, and z_0 is a fixed point of f. By choosing sufficiently small V, we can suppose that $f(V) \subset U$. We know that $(f \circ \pi)_*\pi_1(\widetilde{M}, \widetilde{z}_1) = i_*\pi_1(\dot{U}, f(z_1)) = \pi_*\pi_1(\widetilde{M}, h^{-1}(f(z_1)))$, then we deduce by Theorem 2.5 of [Spa66] that there is a lift $\widetilde{f}$ of f to $\widetilde{M}$ that sends $\widetilde{z}_1$ to $h^{-1}(f(z_1))$. This map $\widetilde{f}$ is a homeomorphism because $\widetilde{f}_*\pi_1(\widetilde{M}, \widetilde{z}_1) = \pi_1(\widetilde{M}, h^{-1}(f(z_1))$ (see Corollary 2.7 in [Spa66]). Moreover, $\widetilde{f}$ can be extend continuously to a homeomorphism of $\widetilde{M} \cup \{\star\}$ that fixes $\star$.

In particular, if f is isotopic to the identity, and if $I = (f_t)_{t\in[0,1]}$ is an identity isotopy of f fixing z_0, then there exists a lift $\widetilde{f}_\cdot(\cdot) : I \times \widetilde{M} \to \widetilde{M}$ of the continuous map $(t, \widetilde{z}) \mapsto f_t(\pi(\widetilde{z}))$ such that $\widetilde{f}_0$ is equal to the identity, because π is a covering projection. Moreover, by choosing V small enough, we know that $\widetilde{f}_t|_{\widetilde{\dot{V}}}$ is conjugate to $f_t|_{\dot{V}}$ for $t \in [0, 1]$, where $\widetilde{\dot{V}}$ is the component of $\pi^{-1}(\dot{V})$ containing $\widetilde{z}_1$. Then $(\widetilde{f}_t)_*\pi_1(\widetilde{M}, \widetilde{z}_1) = \pi_1(\widetilde{M}, h^{-1}(f_t(z_1)),$ therefore $\widetilde{f}_t$ is a homeomorphism by Corollary 2.7 in [Spa66]. We have indeed lifted I to an identity isotopy $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$. Moreover, $\widetilde{f}_t$ can be extended continuously to a homeomorphism of $\widetilde{M} \cup \{\star\}$ that fixes $\star$, and we get an isotopy on $\widetilde{M} \cup \{\star\}$ that fixes $\star$. We still denote by $\widetilde{f}_t$ the homeomorphism of $\widetilde{M} \cup \{\star\}$ and by $\widetilde{I}$ the identity isotopy on $\widetilde{M} \cup \{\star\}$ when there is no ambiguity. We call $\widetilde{I}$ the natural lift of I to $\widetilde{M} \cup \{\star\}$, and $\widetilde{f} = \widetilde{f}_1$ the lift of f to $\widetilde{M} \cup \{\star\}$ associated to I.

Moreover, if I is a maximal isotopy, $\widetilde{f}$ has no contractible fixed point associated to $\widetilde{I}$ on $\widetilde{M}$ and $\widetilde{I}$ is also a maximal isotopy. Recall that $\pi_*\pi_1(\widetilde{M}, \widetilde{z}_1) = i_*\pi_1(\dot{U}, z_1)$. So, $\pi(O)$ is a periodic orbit of type (p, q) associated to I at z_0 for all periodic orbit O of type (p, q) associated to $\widetilde{I}$ at $\star$, where $\frac{p}{q} \in \mathbb{Q}$ is irreducible.

Let $\mathcal{F}$ be an oriented foliation on M_0 such that z_0 is a sink (resp. source). Then there exists a lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}|_{M_0\setminus\{z_0\}}$ to $\widetilde{M}$, and $\star$ is a sink (resp. source) of $\widetilde{\mathcal{F}}$. Denote by W the attracting (resp. repelling) basin of z_0 for $\mathcal{F}$, and by $\widetilde{W}$ the attracting (resp. repelling) basin of $\star$ for $\widetilde{\mathcal{F}}$. Write $\dot{W} = W \setminus \{z_0\}$, $\dot{\widetilde{W}} = \widetilde{W} \setminus \{\star\}$. Let $\widetilde{z}_1 \in \dot{\widetilde{W}}$ be a point sufficient close to $\star$. Then $(\pi|_{\dot{\widetilde{W}}})_*\pi_1(\dot{\widetilde{W}}, \widetilde{z}_1) = \pi_1(\dot{W}, \pi(z_1))$, and hence $\pi|_{\dot{\widetilde{W}}}$ is a homeomorphism between $\dot{\widetilde{W}}$ and $\dot{W}$ by Corollary 2.7 in [Spa66], and can be extended continuously to a homeomorphism between $\widetilde{W}$ and W.

2.14 Extend lifts of a homeomorphism to the boundary

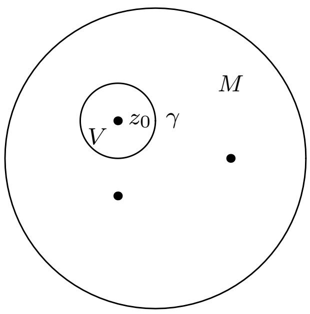
In this section, let M be a plane, f be an orientation preserving homeomorphism of M, and X be an invariant, discrete, closed subset of M with at least 2 points.

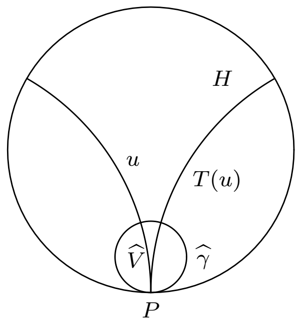
We consider the Poincaré's disk model for the hyperbolic plane H, in which model, H is identified with the interior of the unit disk and the geodesics are segments of Euclidean circles and straight lines that meet the boundary perpendicularly. A choice of hyperbolic structure on $M \setminus X$ provides an identification of the universal cover of $M \setminus X$ with H. A detailed description of the hyperbolic structures can be found in [CB88]. The compactification of the interior of the unit disk by the unit circle induces a compactification of H by the circle S_∞. Let $\pi : H \to M \setminus X$ be the universal cover. Then, $f|_{M\setminus X}$ can be lifted to homeomorphisms of H. Moreover, we have the following result:

Proposition 2.29. *[Han99] Each lift $\widehat{f}$ of $f|_{M\setminus X}$ extends uniquely to a homeomorphism of $H \cup S_\infty$.*

Remark 2.30. When X has infinitely many points, Michael Handel gave a proof in Section 3 of [Han99]; when X has finitely many points, the situation is easier and Handel's proof still works.

In particular, suppose that z_0 is an isolated point in X and is a fixed point of f. Let γ be a sufficiently small circle near z_0 whose lifts to H are horocycles. Fix one lift $\widehat{\gamma}$ of γ. Denote by P the end point of $\widehat{\gamma}$ in S_∞. Fix $z_1 \in \gamma$ and a lift $\widehat{z}_1$ of z_1 in $\widehat{\gamma}$. Let Γ be the group of parabolic covering translations that fix $\widehat{\gamma}$, and T be the parabolic covering translations that generates Γ. Then, π descends to an annulus cover $\pi' : (H/\Gamma, \widetilde{z}_1) \to (M \setminus X, z_1)$, where $\widetilde{z}_1 = \{T^n(\widehat{z}_1) : n \in \mathbb{Z}\}$. Also, $\widehat{z} \mapsto \{T^n(\widehat{z}) : n \in \mathbb{Z}\}$ defines a universal cover $\pi'' : H \to H/\Gamma$.





Let V be the disk containing z_0 and bounded by γ, $\widehat{V}$ be the disk bounded by $\widehat{\gamma}$ which is a component of $\pi^{-1}(V \setminus \{z_0\})$. We know that $\pi''(\widehat{V})$ is an annulus with $\pi''(\widehat{\gamma})$ as one of its boundary. We add a point $\star$ at the other end, and get a disk $\widetilde{V} = \pi''(\widehat{V}) \cup \{\star\}$. As in the previous section, $\pi'|_{\pi''(\widehat{V})}$ extends continuously to a homeomorphism between $\widetilde{V}$ and V, and f can be lifted to a homeomorphism $\widetilde{f}$ of $H/\Gamma \cup \{\star\}$ fixing $\star$. Let $\widehat{f}$ be a lift of $\widetilde{f}|_{H/\Gamma}$ to H, it is also a lift of $f|_{M\setminus X}$ and satisfies $\widehat{f} \circ T = T \circ \widehat{f}$. Moreover, both $\widehat{f}$ and T extend continuously to homeomorphisms of $H \cup S_\infty$ fixing P. We denote still by $\widehat{f}$ and T the two extensions respectively. The formula $\widehat{f} \circ T = T \circ \widehat{f}$ is still satisfied. So, $\widehat{f}|_{H\cup S_\infty \setminus \{P\}}$ descends to a homeomorphism of $(H \cup S_\infty \setminus \{P\})/\Gamma$. Because $(H \cup S_\infty \setminus \{P\})/\Gamma$ is homeomorphic to a compactification of $H/\Gamma \cup \{\star\}$ by adding a circle at infinity S_∞, one knows that $\widetilde{f}$ extends continuously to a homeomorphism of $H/\Gamma \cup \{\star\} \cup S_\infty$.

2.15 The linking number

Let f be an orientation preserving homeomorphism on $\mathbb{R}^2$, and $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. If z_0, z_1 are two fixed points of f, the map

$$t \mapsto \frac{f_t(z_0) - f_t(z_1)}{\|f_t(z_0) - f_t(z_1)\|}$$

descends to a continuous map from $[0,1]/_{0\sim1}$ to S^1. We define the *linking number* between z_0 and z_1 associated to I to be the Brouwer degree of this map, and denote it by $L(I, z_0, z_1)$. We say that z_0 and z_1 are *linked* (relatively to I) if the linking number is not zero.

Suppose that I and I' are identity isotopies of f, and that z_0, z_1 are two fixed points of f. Note the following facts:

- if I and I' fixes z_0 and satisfies $I' \sim J_{z_0}^k I$ as local isotopies at z_0, then one can deduce

$$L(I', z_0, z_1) = L(I, z_0, z_1) + k;$$

- if both I and I' can be viewed as local isotopies at ∞, and if I is equivalent to I' as local isotopies at ∞, then one can deduce

$$L(I', z_0, z_1) = L(I, z_0, z_1).$$

Chapter 3

Dynamics near an isolated fixed points with index one and zero rotation

3.1 Proof of the main theorem

Let M be an oriented surface, $f : M \to M$ be an area preserving homeomorphism of M isotopic to the identity, and z_0 be an isolated fixed point of f such that $i(f, z_0) = 1$. Let I be an identity isotopy of f fixing z_0 such that its rotation set, which was defined in section 2.10, is reduced to an integer k.

We say that the property **P)** holds for (f, I, z_0) if there exists $\varepsilon > 0$, such that either for all irreducible $p/q \in (k, k+\varepsilon)$, or for all irreducible $p/q \in (k-\varepsilon, k)$, there exists a contractible periodic orbit $O_{p/q}$ of type (p, q) associated to I at z_0, such that $\mu_{O_{p/q}} \to \delta_{z_0}$ as $p/q \to k$, in the weak-star topology, where $\mu_{O_{p/q}}$ is the invariant probability measure supported on $O_{p/q}$,

Our aim of this section is to prove the following result:

Theorem 3.1 (Theorem 1.1)**.** *Under the previous assumptions, if one of the following situation occurs,*

a) M is a plane, f has only one fixed point z_0 and has a periodic orbit besides z_0;

b) the total area of M is finite,

*then the property **P)** holds for (f, I, z_0).*

Remark 3.2. Let I' be a local isotopy of f at z_0 such that $\rho_s(I', z_0)$ is reduced to 0. Since f is area preserving and $i(f, z_0) = 1$, by Proposition 2.15, I' has either a positive or a negative rotation type. Let $\mathcal{F}'$ be a locally transverse foliation of I'. If I' has a positive rotation type, then z_0 is a sink of $\mathcal{F}'$ and the interval in Property **P)** is $(k, k+\varepsilon)$; if I' has a negative rotation type, then z_0 is a source and the interval in Property **P)** is $(k-\varepsilon, k)$.

We assume that I' has a positive rotation type in this section, the other case can be treated similarly.

Remark 3.3. If z_0 is not accumulated by periodic orbits, since the rotation set is reduced to an integer and $i(f, z_0) = 1$, z_0 is an indifferent fixed point by Proposition 2.23. Then, by the assertion viii) of Proposition 2.20, one deduces that $\rho(I, z_0)$ is equal to this integer.

We will prove the theorem in several cases.

3.1.1 The case where M is a plane

In this section, we suppose that M is a plane, and that I is a maximal identity isotopy of f such that $\mathrm{Fix}(I)$ is reduced to z_0. We will prove the following result in this section and get the proof of the first part of Theorem 1.1 as a corollary.

Theorem 3.4. *Under the previous assumption, if $\rho_s(I,z_0)$ is reduced to 0, and if f has another periodic orbit besides z_0, then the property **P)** holds for (f,I,z_0).*

This result is an important one in the proof of Theorem 1.1. In the latter cases, we will always reduce the problem to this case and get the result as a corollary. Before proving this result, we first prove the first case of Theorem 1.1 as a corollary.

Proof of the first case of Theorem 1.1. If k is equal to 0, the result follows directly from Theorem 3.4. So, we only deal with the case where $\rho_s(I,z_0)$ is reduced to a non-zero integer k. Let J be an identity isotopy of the identity fixing z_0 such that the blow-up rotation number $\rho(J,z_0)$ is equal to 1. Write $I'=J^{-k}I$. By the first assertion of Proposition 2.20, $\rho_s(I',z_0)$ is reduced to 0. Since f has exactly one fixed point, I' is maximal and the property **P)** holds for (f,I',z_0). A periodic orbit in the annulus $M\setminus\{z_0\}$ with rotation number p/q associated to I' is a periodic orbit with rotation number $k+p/q$ associated to I. Therefore, the property **P)** holds for (f,I,z_0). □

Now we begin the proof of Theorem 3.4 by some lemmas.

Lemma 3.5. *Let g be a homeomorphism of $\mathbb{R}^2$, I' be a maximal identity isotopy of g that fixes z_0, and $\mathcal{F}'$ be a transverse foliation of I'. Suppose that z_0 is an isolated fixed point of g and a sink of $\mathcal{F}'$. Let W' be the attacting basin of z_0 for $\mathcal{F}'$. Suppose that either W' is equal to $\mathbb{R}^2$ or W' is a proper subset of $\mathbb{R}^2$ whose boundary is the union of some proper leaves of $\mathcal{F}'$. Let U be a Jordan domain containing z_0 that satisfies $U\subset W'$ and $g(U)\subset W'$.*

If there exist a compact subset $K\subset U$ and $\varepsilon>0$ such that K contains a q-periodic orbit $O_{p/q}$ with rotation number p/q in the annulus $\mathbb{R}^2\setminus\{z_0\}$ for all irreducible $p/q\in(0,\varepsilon)$, then $\mu_{O_{p/q}}$ converges, in the weak-star topology, to the Dirac measure δ_{z_0} as $p/q\to 0$, where $\mu_{O_{p/q}}$ is the invariant probability measure supported on $O_{p/q}$.

Proof. We only need to prove that for every continuous function $\varphi: W'\to\mathbb{R}$, for every $\eta>0$, there exists $\delta>0$, such that for every $q-$periodic orbit $O\subset K$ with irreducible rotation number $p/q<\delta$, we have

$$|\int\varphi d\mu_O-\varphi(z_0)|<\eta,$$

where μ_O is the invariant probability measure supported on O.

Let V be a neighborhood of z_0 such that $|\varphi(z)-\varphi(z_0)|<\eta/2$ for all $z\in V$. Let $\pi:\mathbb{R}\times(-\infty,0)\to W'\setminus\{z_0\}$ be the universal cover which sends the vertical lines upward to the leaves of $\mathcal{F}'$, and $p_1:\mathbb{R}\times(-\infty,0)\to\mathbb{R}$ be the projection onto the first coordinate. Let $\widetilde{U}=\pi^{-1}(U\setminus\{z_0\})$, $\widetilde{K}=\pi^{-1}(K\setminus\{z_0\})$ and $\widetilde{g}$ be the lift of g to $\widetilde{U}$ associated to I'. By the assumptions about W', we know that any arc that is positively transverse to $\mathcal{F}'$ cannot come back into W' once it leaves W'. So

$$p_1(\widetilde{g}(z))-p_1(z)>0,\ \text{for all } z\in\widetilde{K}.$$

Therefore there exists $\eta_1>0$ such that for all $z\in\pi^{-1}(K\setminus V)$, one has

$$p_1(\widetilde{g}(z))-p_1(z)>\eta_1.$$

One deduces that for all $q-$periodic orbit $O \subset K$ with irreducible rotation number $p/q < \delta = \frac{\eta\eta_1}{4|\sup_K \varphi|}$,

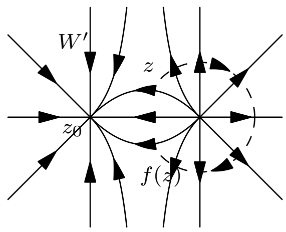
$$\frac{\#(O \setminus V)\eta_1}{q} < \frac{p}{q},$$

hence,

$$|\int \varphi d\mu_O - \varphi(z_0)| < \eta/2 + 2\sup_K |\varphi| \frac{\#(O \setminus V)}{q} < \eta.$$

□

Remark 3.6. In this lemma, the homeomorphism g do not need to be area preserving. The assumptions about W' prohibit the following bad situation:



Lemma 3.7. *If $\rho_s(I, z_0)$ is reduced to 0, and if f can be blown-up at ∞ such that the blow-up rotation number at ∞, that is defined in Section 2.10, is different from 0, then the property **P)** holds for (f, I, z_0).*

In order to to prove this lemma, we need the following sublemma:

Sublemma 3.8. *Under the conditions of the previous Lemma, $\rho(I, \infty)$ is negative, and there exists $\varepsilon > 0$ such that for all irreducible $p/q \in (0, \varepsilon)$, there exists a q-periodic orbit with rotation number p/q in the annulus $M \setminus \{z_0\}$.*

Proof. We consider a transverse foliation $\mathcal{F}$ of I. It has exactly two singularities z_0 and ∞. Since f is area preserving, $f|_{M\setminus\{z_0\}}$ satisfies the intersection property, and using the remark that follows Proposition 2.12, one can deduce that $\mathcal{F}$ does not have any closed leaf. Because f can be blown-up at ∞ and the blow-up rotation number $\rho(I, \infty)$ is different from 0, we deduce that ∞ is either a sink or a source. By the assumption in Remark 3.2, z_0 is a sink of $\mathcal{F}$, so ∞ is a source of $\mathcal{F}$, and hence $\rho(I, \infty)$ is smaller than 0. Write $\rho = -\rho(I, \infty)$. We denote by S_∞ the circle added at ∞ when blowing-up f at ∞, and still by f the extension of f to $M \sqcup S_\infty$. One has to consider the following two cases:

- Suppose that z_0 is accumulated by periodic orbits. Let z_1 be a periodic point of f in the annulus $M \setminus \{z_0\}$. Its rotation number is strictly positive. We denote by ε this number. Because the rotation set $\rho_s(I, z_0)$ is equal to 0, the rotation number of a periodic orbit tends to 0 as the periodic orbit tend to z_0. Hence for all irreducible $p/q \in (0, \varepsilon)$, there exists a periodic orbit near z_0 with rotation number $r/s \in (0, p/q)$. The restriction of the homeomorphism f to the annulus $M \setminus \{z_0\}$ satisfies the intersection property, then by Proposition 2.25, there exists a q-periodic orbit with rotation number p/q in the annulus for all irreducible $p/q \in (0, \varepsilon)$.
- Suppose that z_0 is not accumulated by periodic orbits. Then, by Proposition 2.23 z_0 is an indifferent fixed point, and by the assertion viii) of Proposition 2.20, $\rho(I, z_0)$, which was defined in Section 2.9, is equal to 0. Let K_0 be a small enough invariant continuum at z_0 such that $\rho(I, K_0) = 0$ (see Section 2.9). We denote by

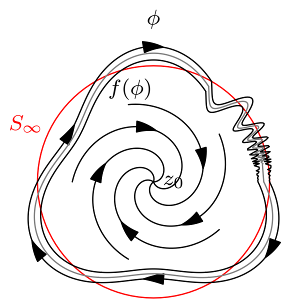
$(M\setminus K_0)\sqcup\mathbb{T}^1\sqcup S_\infty$ the prime-ends compactification at the ends K_0 and the compactification at ∞, which is a closed annulus. We can extend f to both boundaries and get a homeomorphism of the closed annulus satisfying the intersection condition. Moreover, the rotation number of f on the upper boundary $\mathbb{T}^1$ is equal to 0, and on the lower boundary S_∞ is equal to ρ. So, by Proposition 2.24, for all irreducible p/q between 0 and ρ, there exists a periodic orbit in the annulus with rotation number p/q.

□

Remark 3.9. In the first case of the proof, it is natural to think that we can prove by a generalization of Poincaré-Birkhoff theorem that there exists a periodic orbit in the annulus with rotation number p/q for all irreducible p/q between 0 and ρ. But in fact, the annulus in this case is half-open, and we do not know whether there exists such a generalization of Poincaré-Birkhoff theorem. So, we choose another periodic orbit to avoid treating the half-open annulus.

Proof of Lemma 3.7. Let us blow-up f at ∞ by adding a circle S_∞ and paste two copies of the closed disk by S_∞. We get a sphere S and a homeomorphism f' that equals to f on each copy and has two fixed points z_0 and $\sigma(z_0)$, where σ is the natural involution. Let I' be an identity isotopy that fixes z_0 and $\sigma(z_0)$ and satisfies $\rho_s(I',z_0)=\{0\}$. Because I is a maximal isotopy, $f|_{M\setminus\{z_0\}}$ has no contractible fixed point associated to I. Because the blow-up rotation number $\rho(I,\infty)$ is different from 0, the extension of f to S_∞ does not have any fixed point with rotation number 0 (associated to I). So, $f'|_{S\setminus\{z_0,\sigma(z_0)\}}$ has no contractible fixed point associated to $I'|_{S\setminus\{z_0,\sigma(z_0)\}}$. Therefore, I' is a maximal isotopy, and one knows $\mathrm{Fix}(I')=\{z_0,\sigma(z_0)\}$. Let $\mathcal{F}'$ be a transverse foliation of I'. Then $\mathcal{F}'$ has exactly two singularities z_0 and $\sigma(z_0)$. By the assumption in Remark 3.2, z_0 is a sink of $\mathcal{F}'$. Since the involution σ is orientation reversing, $\rho(I',\sigma(z_0))=0$ and I' has a negative rotation type at $\sigma(z_0)$. So $\sigma(z_0)$ is a source of $\mathcal{F}'$, and hence $\mathcal{F}'$ does not have any petal. One has to consider the following two cases:

- Suppose that all the leaves of $\mathcal{F}'$ are curves from $\sigma(z_0)$ to z_0. Sublemma 3.8 implies that the compact set $M\sqcup S_\infty$ satisfies the conditions of Lemma 3.5, and we can deduce the result.
- Suppose that there exists a closed leaf in $\mathcal{F}'$. Since f is area preserving, similarly to the remark that follows Proposition 2.12, one can deduce that there does not exist any closed leaf in M or in $\sigma(M)$. So, each closed leaf intersects S_∞. Let W' be the basin of z_0 for $\mathcal{F}'$. Then $\partial W'$ is a closed leaf, and hence intersects S_∞. Denote this leaf by ϕ. We suppose that z_0 is to the right of ϕ, the other case can be treated similarly. Denote by $R(\phi)$ (resp. $L(\phi)$) the component of $S\setminus\phi$ to



the right (resp. left) of ϕ. Since $f'(\phi)$ is included in $R(\phi)$, we know that both $R(\phi)\cap M$ and $R(\phi)\setminus(M\cup S_\infty)$ are not empty. Choose a suitable essential curve

$\Gamma \subset (R(\phi) \cap L(f(\phi))) \subset W'$ that transversely intersects S_∞ at only finitely many points (see the gray curve between ϕ and $f(\phi)$ in the picture). Then, $(L(\Gamma) \cap M)$ has finitely many connected components, and so does $(L(\Gamma) \cap (M \cup S_\infty))$. Moreover, each component of $(L(\Gamma) \cap (M \cup S_\infty))$ contains a segment of S_∞.
Since both M and S_∞ are invariant by f', one knows that $f'^{-1}(L(\Gamma) \cap (M \cup S_\infty))$ is included in $L(\Gamma) \cap (M \cup S_\infty)$. So, if V is a component of $(L(\Gamma) \cap (M \cup S_\infty))$, there exists $n > 0$ such that $f'^{-n}(V) \subset V$. Moreover, one knows that $f'^{-n}(V \cap S_\infty) \subset V \cap S_\infty$ and that the rotation number of each point in S_∞ is equal to ρ, so there exists $m > 0$ such that $\rho = m/n$ and the rotation number of every periodic point of f' in V is equal to ρ. Therefore, the rotation number of every periodic point $z \in (L(\Gamma) \cap M)$ of f' is equal to the rotation number of S_∞. So, all the periodic orbits in the annulus $M \setminus \{z_0\}$ with rotation number in $(0, \rho)$ is contained in $R(\Gamma) \cap M$. We find a compact set $\overline{R(\Gamma) \cap M}$ that satisfies the conditions of Lemma 3.5, and can deduce the result. □

Lemma 3.10. *If $\rho_s(I, z_0)$ is reduced to 0, and if $O \subset M \setminus \{z_0\}$ is a periodic orbit of f, then the rotation number of O (associated to I) is positive.*

Proof. Let $\mathcal{F}$ be a transverse foliation of I. Then $\mathcal{F}$ has only one singularity z_0, and z_0 is a sink of $\mathcal{F}$ by the assumption in Remark 3.2. Since f is area preserving, by the remark that follows 2.12 one knows that $\mathcal{F}$ does not have any closed leaf. Let W be the attracting basin of z_0 for $\mathcal{F}$. It is either M or a proper subset of M whose boundary is the union of some proper leaves. In the first case, any periodic orbit of $f|_{M\setminus\{z_0\}}$ has a positive rotation number associated to I, and the proof is finished. In the second case, note that each connected component of $M \setminus \overline{W}$ is a disk foliated by proper leaves, and hence does not contain any loop that is transverse to $\mathcal{F}$. Moreover, any loop transverse to $\mathcal{F}$ can not meet a boundary leaf of W, and hence is contained in W. One deduces that every periodic orbit of f distinct from $\{z_0\}$ is contained in W, and its trajectory along the isotopy is homotopic to a transverse loop in W. So, its rotation number is positive. □

Lemma 3.11. *If $\rho_s(I, z_0)$ is reduced to 0, and if f has another periodic orbit besides z_0, then there exist an interger $q > 1$ and a q-periodic orbit O with rotation number $1/q$ (associated to I) such that $f|_{M\setminus(O\cup\{z_0\})}$ is isotopic to a homeomorphism $R_{1/q}$ satisfying $R_{1/q}^q = \mathrm{Id}$.*

Proof. Let O_0 be a periodic orbit of f distinct from $\{z_0\}$. By the previous lemma, the rotation number ρ of O_0 in the annulus $M \setminus \{z_0\}$ associated to I is positive. Similarly to the proof of Sublemma 3.8, there exists a q-periodic orbit with rotation number p/q in the annulus $M \setminus \{z_0\}$ for all irreducible $p/q \in (0, \rho)$. Let $\mathcal{F}$ be a transverse foliation of I. One knows that z_0 is a sink of $\mathcal{F}$ by the assumption in Remark 3.2. Let W be the attracting basin of z_0 for $\mathcal{F}$. One has to consider the following three cases:

i) Suppose that W is equal to M.
Let $T : (x, y) \mapsto (x + 1, y)$ be the translation of $\mathbb{R}^2$. It induces a universal covering map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/T \simeq \mathbb{T}^1 \times \mathbb{R}$. Let $h : M \setminus \{z_0\} \to \mathbb{T}^1 \times \mathbb{R}$ be an orientation preserving map that maps the leaves of $\mathcal{F}$ to the vertical lines $\{\pi(\{x\} \times \mathbb{R}) : x \in \mathbb{R}\}$ upward. Write $I' = (h \circ f_t \circ h^{-1})_{t\in[0,1]}$, and $f' = h \circ f \circ h^{-1}$. We will prove that there exists a positive integer q, and a q-periodic orbit O of f' with rotation number $1/q$ (associated to I') such that $f'|_{(\mathbb{T}^1\times\mathbb{R})\setminus O}$ is isotopic to a homeomorphism $R_{1/q}$ satisfying that $R_{1/q}^q = \mathrm{Id}$, and hence $h^{-1}(O)$ is a q-periodic orbit of f with rotation number $1/q$ (associated to I) such that $f|_{M\setminus(h^{-1}(O)\cup\{z_0\})}$ is isotopic to $h^{-1} \circ R_{1/q} \circ h$.

Fix a q-periodic orbit O of f' with rotation number $1/q$ in the annulus $M\setminus\{z_0\}$ for $1/q\in(0,\rho)$. Choose $0<M_1<M_2$ such that

$$O\subset\mathbb{T}^1\times(-M_1,M_1),\quad \text{and } (\bigcup_{t\in[0,1]} f'_t(\mathbb{T}^1\times[-M_1,M_1]))\subset\mathbb{T}^1\times(-M_2,M_2).$$

Let $\widetilde{f}$ be the lift of f' associated to I'. One knows that

$$p_1(\widetilde{f}(\widetilde{z}))-p_1(\widetilde{z})>0\quad\text{for all } \widetilde{z}\in\mathbb{R}^2,$$

where p_1 is the projection to the first factor. Let φ_1 be the homeomorphism of $\mathbb{T}^1\times\mathbb{R}$ whose lift to $\mathbb{R}^2$ is defined by

$$\widetilde{\varphi}_1(x,y)=\begin{cases}(x,y), & \text{for } |y|\le M_2,\\ (x+|y|-M_2,y), & \text{for } |y|>M_2.\end{cases}$$

We know that $\eta(y)=\sup_{x\in\mathbb{R}}|p_2(\widetilde{f}'(x,y))-y|$ is a continuous function, where p_2 is the projection onto the second factor. So, there exist $M_3>M_2$ and a homeomorphism φ_2 of $\mathbb{T}^1\times\mathbb{R}$ whose lift $\widetilde{\varphi}_2$ to $\mathbb{R}^2$ satisfies $p_1\circ\widetilde{\varphi}_2=\mathrm{Id}$ and

$$\widetilde{\varphi}_2(x,y)=\begin{cases}(x,y), & \text{for } |y|\le M_2,\\ (x,y+\mathrm{sign}(y)(\eta(y)+1)), & \text{for } |y|\ge M_3.\end{cases}$$

Let $f''=\varphi_2\circ\varphi_1\circ f'$. It is a contraction near each end and hence can be blown-up at each end by adding a circle. Moreover, by choosing suitable blow-up, the rotation numbers at the boundary can be any real number, and we get a homeomorphism $\overline{f''}$ of closed annulus and a lift $\widetilde{f''}$ of $\overline{f''}$ such that O is a $(1,q)$-periodic orbit and $\rho(\widetilde{f''})$ (see Section 2.12 for the definition) is a closed subset in $(0,\infty)$. One deduces by Proposition 2.28 that O is topologically monotone (Otherwise $I(1/q)=[0,1/(q-1)]\subset\rho(\widetilde{f''})$). Therefore, $f''|_{(\mathbb{T}^1\times\mathbb{R})\setminus O}$ is isotopic to a homeomorphism $R_{1/q}$ satisfying $R_{1/q}^q=\mathrm{Id}$, and so is $f'|_{(\mathbb{T}^1\times\mathbb{R})\setminus O}$. The lemma is proved.

ii) Suppose that W is a proper subset of M whose boundary is the union of some proper leaves, and that z_0 is not accumulated by periodic orbits.
In this case, one knows by Remark 2.22 that f can be blown-up at ∞, and that the blow-up rotation number $\rho(I,\infty)$ is equal to 0. One knows by Remark 3.3 that z_0 is a non-accumulated indifferent point, and that $\rho(I,z_0)$ is equal to 0.
Recall that there exists a q-periodic orbit with rotation number p/q in the annulus $M\setminus\{z_0\}$ for all irreducible $p/q\in(0,\rho)$. We fix a q-periodic orbit O of f with rotation number $1/q$ in the annulus $M\setminus\{z_0\}$ for $1/q\in(0,\rho)$. Let γ_1 be a simple closed curve that separates O and z_0. Denote by U^- the component of $M\setminus\gamma_1$ containing O. We deduce by the assertions i) and ii) of Proposition 2.20 that there exists a neighborhood of z_0 that does not contain any q-periodic point of f with rotation number $1/q$. So, by choosing γ_1 sufficiently close to z_0, we can suppose that all the q-periodic points of f with rotation number $1/q$ are contained in U^-. Let γ_2 be a simple closed curve that separate γ_1 and z_0 such that $\cup_{t\in[0,1]}f_t(\overline{U^-})$ is in the component of $M\setminus\gamma_2$ containing γ_1. Denote by U the component of $M\setminus\gamma_2$ containing z_0. Let $V\subset U$ be a small Jordan domain containing z_0 such that $\cup_{t\in[0,1]}f_t(\overline{V})\subset U$, and $K\subset V$ be a sufficiently small invariant continuum at z_0 such that $\rho(I,K)=0$. Let $M\setminus K\cup S_\infty\cup S^1$ be a compactification of $M\setminus K$, where S_∞ is the circle added when blowing f at ∞ and S^1 is the circle added when blowing $f|_{M\setminus K}$ at the end K. It is a closed annulus, and $f|_{M\setminus K}$ extends continuously to a homeomorphism $\overline{f}$ of

$M \setminus K \cup S_\infty \cup S^1$. The homeomorphism $\overline{f}$ has a $(1, q)$ periodic orbit O and hence by Proposition 2.28 has a $(1, q)$ topologically monotone periodic orbit O' (It could be equal to O or different from O). Since the rotation number of $\overline{f}$ at both boundary is equal to 0, O' is included in $M \setminus K$ and hence in U^-. So, $f|_{M\setminus(K\cup O')}$ is isotopic to a homeomorphism $R_{1/q}$ satisfying that $R_{1/q}^q = \mathrm{Id}$.

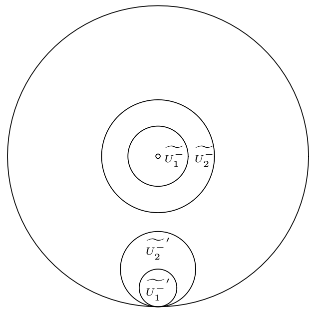
Let $h : M \setminus K \to M \setminus \{z_0\}$ be a homeomorphism whose restriction to $M \setminus U$ is equal to the identity. Then, $f' = h \circ f|_{M\setminus K} \circ h^{-1}$ is a homeomorphism of $M \setminus \{z_0\}$ which coincides f in $M\setminus U$. The restriction of $f' \circ f^{-1}$ to $M\setminus U$ is equal to the identity, using Alexander's trick one deduces that $f' \circ f^{-1}|_{M\setminus(O'\cup\{z_0\})}$ is isotopic to the identity. So, $f'|_{M\setminus(O'\cup\{z_0\})}$ and $f|_{M\setminus(O'\cup\{z_0\})}$ are isotopic. Therefore, $f|_{M\setminus(O'\cup\{z_0\})}$ is isotopic to $R'_{1/q} = h|_{M\setminus(O'\cup\{z_0\})} \circ R_{1/q} \circ h^{-1}|_{M\setminus(O'\cup\{z_0\})}$ which satisfies of course $R'^q_{1/q} = \mathrm{Id}$.

iii) Suppose that W is a proper subset of M whose boundary is the union of some proper leaves, and that z_0 is accumulated by periodic orbits.

As in case ii), f can be blown-up at ∞ and the blow-up rotation number $\rho(I, \infty)$ is equal to 0. Recall that there exists a q-periodic orbit with rotation number p/q in the annulus $M\setminus\{z_0\}$ for all irreducible $p/q \in (0, \rho)$. Fix two prime integers q_1 and q_2 such that $1/q_2 < 1/q_1 < \rho$. Choose a q_1-periodic orbit O_1 and a q_2-periodic orbit O_2 in $M\setminus\{z_0\}$ with rotation number (associated to I) $1/q_1$ and $1/q_2$ respectively. Recall that the rotation number of every periodic orbit in $M \setminus \{z_0\}$ is positive and that $\rho_s(I, z_0)$ is reduced to 0. One deduces by the assertion i) and ii) that for any given integer $q > 1$, there is a neighborhood of z_0 that does not contain any q-periodic point of f. So, there exists a Jordan domain U containing z_0 that does not contain any periodic point of f with period not bigger that q_2 except z_0. Let $\gamma_1 \subset U$ be a simple closed curve that separates z_0 and $O_1 \cup O_2$. Denote by U_1^- the component of $M \setminus \gamma_1$ containing $O_1 \cup O_2$. Let γ_2 be a simple closed curve that separates γ_1 and z_0 such that the trajectory of each $z \in \overline{U_1^-}$ along I^{q_2} is in the component U_2^- of $M \setminus \gamma_2$ containing γ_1. Let γ_3 be a simple closed curve that separates γ_2 and z_0 such that the trajectory of each $z \in \overline{U_2^-}$ along I^{q_2} is in the component U_3^- of $M \setminus \gamma_3$ containing γ_2. Since $\gamma_3 \subset U$, there does not exist any periodic points of f with periodic not bigger that q_2 in γ_3. We can perturb f in $M\setminus(\overline{U_3^-} \cup \{z_0\})$ and get a homeomorphism f' such that f' has finitely many periodic points with periods not bigger than q_2 in $M \setminus U_3^-$.

Let X be the union of periodic orbits of f' with periodic not bigger than q_2 that intersects $M \setminus \overline{U_3^-}$. It is a finite set containing z_0. We consider the annulus covering $\pi : \widetilde{M} \to M\setminus X$ such that the restriction of π to a sufficiently small annulus near one end is a homeomorphism between this annulus and a small annulus near ∞ in $M\setminus X$. As in Section 2.13, we add a point $\star$ at this end of $\widetilde{M}$. Let $\widetilde{U_i^-}$ be the component of $\pi^{-1}(U_i^-)$ that has an end $\star$ and $\widetilde{O}'_i$ be the lift of O'_i in $\widetilde{U_2^-}$ for $i = 1, 2$. Let $\widetilde{f}'$ be the lift of $f'|_{M\setminus X}$. It extends continuously to a homeomorphism of $\widetilde{M} \cup \{\star\}$, and the dynamics of $\widetilde{f}'$ near $\star$ is conjugate to the dynamics of f' near ∞. So, $\widetilde{f}'$ can be blown-up at $\star$, and by choosing a suitable isotopy $\widetilde{I}'$ of $\widetilde{f}'$, the blow-up rotation number $\rho(\widetilde{I}', \star)$ is equal to 0. Moreover, $\widetilde{O}'_i$ is a q_i-periodic orbit of $\widetilde{f}'$ with rotation number $1/q_i$ (associated to $\widetilde{I}'$), for $i = 1, 2$. Referring to Section 2.14, one knows that $\widetilde{f}'$ can be blown-up at the other end.

We blow-up $\widetilde{f}$ at both ends and get a homeomorphism $\overline{\widetilde{f}'}$ of a closed annulus. For $i = 1, 2$, the homeomorphism $\overline{\widetilde{f}'}$ has a $(1, q_i)$ periodic orbit, so one can deduce by Proposition 2.28 that $\overline{\widetilde{f}'}$ has a $(1, q_i)$ topologically monotone periodic orbit $\widetilde{O}''_i$. The circle we added at $\star$ does not contain any periodic points with rotation number different from 0, so it does not contain $\widetilde{O}''_1$ or $\widetilde{O}''_2$. The rotation number of $\widetilde{\widetilde{f}}'$ at the circle we added at the other end is different from $1/q_1$ or $1/q_2$. Suppose that it is different from $1/q_1$, the other case can be treated similarly. Then, $\widetilde{O}''_1$ is included in $\widetilde{M}$, and $\pi(\widetilde{O}''_1)$ is a periodic orbit of f' of period not bigger than q_1. So, $\pi(\widetilde{O}''_1)$ is included in $\overline{U_3^-}$, and hence is a periodic orbit of f in U_1^-.



We will prove by contradiction that $\widetilde{O}''_1$ is included in $\widetilde{U_1^-}$. Otherwise, suppose that there exists $\widetilde{z} \in \widetilde{O}''_1$ in another component $\widetilde{U_1^-}'$ of $\pi^{-1}(U_1^-)$. Then, $\widetilde{z}$ is a fixed point of $\widetilde{f}^{q_1}$. Let $\widetilde{U_2^-}'$ be the component of $\pi^{-1}(U_2^-)$ containing $\widetilde{U_1^-}'$. Since $f'^{q_1}(U_1^-) \subset U_2^-$, one deduces that $\widetilde{f}'^{q_1}(\widetilde{U_1^-}') \subset \widetilde{U_2^-}'$. Recall that the rotation number of $\widetilde{O}''_1$ is $1/q_1$. So, the rotation number of $\widetilde{\widetilde{f}}'$ at the outer boundary is $1/q_1$, which contradicts our assumption.

Let $h : \widetilde{M} \to M \setminus \{z_0\}$ be a homeomorphism whose restriction to $\widetilde{U_2^-}$ is equal to π. As in the end of case ii), we deduce that $f|_{M\setminus(\pi(\widetilde{O}''_1)\cup\{z_0\})}$ is isotopic to a homeomorphism R_{1/q_1} satisfying that $R_{1/q_1}^{q_1} = \mathrm{Id}$. The lemma is proved.

$\square$

Proof of Theorem 3.4. By the previous lemma, there exist an integer $q' > 1$ and a q'-periodic orbit O with rotation number $1/q'$ (associated to I) such that $f|_{M\setminus(O\cup\{z_0\})}$ is isotopic to a homeomorphism $R_{1/q'}$ satisfying $R_{1/q'}^{q'} = \mathrm{Id}$. Let $I' = (\varphi_t)_{t\in[0,1]}$ be an identity isotopy of $f^{q'}$ that fixes every point in $O \cup \{z_0\}$. Since the rotation number of O associated to I is $1/q'$, each point in O is a fixed point of $f^{q'}$ and its rotation number associated to $I^{q'}$ is 1. Because I' fixes $O \cup \{z_0\}$, $I'|_{M\setminus\{z_0\}}$ is homotopic to $J_{z_0}^{-1} I^{q'}|_{M\setminus\{z_0\}}$, where J_{z_0} is an identity isotopy of the identity fixing z_0 such that $\rho(J_{z_0}, z_0) = 1$. By the first assertion of Proposition 2.20, one knows that $\rho_s(I', z_0)$ is reduced to -1.

Let $\pi' : \widehat{M} \to M \setminus O$ be the universal cover. Since $M \setminus O$ is a surface of finite type, we can endow it a hyperbolic structure, and $\widehat{M}$ can be viewed to be the hyperbolic plane. Fix $\widehat{z}_0 \in \pi'^{-1}(z_0)$. Let $\widehat{f}$ be the lift of $f|_{M\setminus O}$ that fixes $\widehat{z}_0$. Then, $\widehat{f}$ can be blown-up at ∞.

Let $\widehat{I}' = (\widehat{\varphi}_t)_{t\in[0,1]}$ be the identity isotopy of $\widehat{f}^{q'}$ that lifts I'. Then, $\rho_s(\widehat{I}', \widehat{z}_0)$ is reduced to -1. On the other hand, ∞ is accumulated by the points of $\pi'^{-1}\{z_0\}$ which are fixed points of $\widehat{I}'$, so by the assertion ii) of Proposition 2.20, one knows that 0 is belong to $\rho_s(\widehat{I}', \infty)$. But $\widehat{f}$ can be blown-up at ∞, by the assertion vii) of Proposition 2.20, we know that $\rho_s(\widehat{I}', \infty)$ is reduced to 0.

Let $\widehat{I}_0$ be an identity isotopy of $\widehat{f}$ that fixes $\widehat{z}_0$ and satisfies $\rho_s(\widehat{I}_0, \widehat{z}_0) = \{0\}$. Then $\rho_s(\widehat{I}_0^{q'}, \widehat{z}_0)$ is reduced to 0, and hence $\widehat{I}_0^{q'}|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ is homotopic to $J_{\widehat{z}_0} \widehat{I}'|_{\widehat{M}\setminus\{\widehat{z}_0\}}$. So, $\rho_s(\widehat{I}_0^{q'}, \infty)$ is reduced to -1, and by the assertion i) of Proposition 2.20, we deduce that $\rho_s(\widehat{I}_0, \infty)$

is reduced to $-1/q'$. Since $\widehat{f}$ can be blown-up at ∞, by the assertion vii) of Proposition 2.20, one knows that the blow-up rotation number $\rho(\widehat{I}_0,\infty)$ is equal to $-1/q'$.

Every $\widehat{z}'_0 \in \pi'^{-1}\{z_0\}\setminus\{\widehat{z}_0\}$ is a contractible fixed point of $\widehat{f}^{q'}|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ associated to $\widehat{I}'|_{\widehat{M}\setminus\{\widehat{z}_0\}}$, so it is not a contractible fixed point of $\widehat{f}|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ associated to $\widehat{I}_0|_{\widehat{M}\setminus\{\widehat{z}_0\}}$.

Let $\widehat{O}'$ be a periodic orbit of $\widehat{f}$ in the annulus $\widehat{M}\setminus\{\widehat{z}_0\}$ such that $z_0\notin\pi'(\widehat{O}')$ and the rotation number of $\widehat{O}'$ associated to $\widehat{I}_0|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ is p/q. Then $\widehat{O}'$ is a periodic orbit of $\widehat{f}^{q'}$ in the annulus $\widehat{M}\setminus\{\widehat{z}_0\}$ and the rotation number associated to $\widehat{I}'|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ is $\frac{pq'}{q}-1$. So, $\pi'(\widehat{O}')$ is a periodic orbit of f in the annulus $M\setminus\{z_0\}$, the rotation number associated to I' is $\frac{pq'}{q}-1$, the rotation number associated to $I^{q'}$ is $\frac{pq'}{q}$, and the rotation number associated to I is p/q. In particular, if $\widehat{z}'$ is a contractible fixed point of $\widehat{f}|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ associated to $\widehat{I}_0|_{\widehat{M}\setminus\{\widehat{z}_0\}}$, $\pi'(\widehat{z}')$ is a contractible fixed point of $f|_{M\setminus\{z_0\}}$ associated to $I|_{M\setminus\{z_0\}}$. So, $\widehat{f}|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ does not have any contractible fixed point associated to $\widehat{I}_0$.

Moreover, if p/q is irreducible, and if $\widehat{O}'$ is a periodic orbit of $\widehat{f}$ of type (p,q) associated to $\widehat{I}_0$ at $\widehat{z}_0$ such that $z_0\notin\pi'(\widehat{O}')$, then $\pi'(\widehat{O}')$ is a periodic orbit of f of type (p,q) associated to I at z_0.

By Lemma 3.7, the property **P)** holds for $(\widehat{f},\widehat{I}_0,\widehat{z}_0)$, and then holds for (f,I,z_0). □

3.1.2 The case where the total area of M is finite

In this section, we assume that the area of M is finite. Recall that f is an area preserving homeomorphism of M, that z_0 is an isolated fixed point of f satisfying $i(f,z_0)=1$, that I is an identity isotopy of f that fixes z_0 and satisfies $\rho_s(I,z_0)=\{k\}$. Let (X,I_X) be a maximal extension of I that satisfies $\rho_s(I_X,z_0)=\rho_s(I,z_0)$. Write $X_0=X\setminus\{z_0\}$. Then, X_0 is a closed subset of $\mathrm{Fix}(f)$, and I_X can be extended to a maximal identity isotopy on $M\setminus X_0$ that fixes z_0. To simplify the notation, we still denote by I_X this extension. Moreover, by definition of Jaulent's preorder, we know that a periodic orbit of type (p,q) associated to I_X at z_0 is a periodic orbit of type (p,q) associated to I at z_0. Let M_0 be the connected component of $M\setminus X_0$ that contains z_0. Of course the total area of M_0 is also finite. When M is a sphere, $f|_{M\setminus\{z_0\}}$ has at least one fixed point (see Section 2.4), and hence X_0 is not empty. So, M_0 is not a sphere. To simplify the notations, we denote by f_0 the restriction of f to M_0, and by I_0 the restriction of I_X to M_0. If the property **P)** holds for (f_0,I_0,z_0), it holds for (f,I,z_0). So, we will prove the following proposition, and the second part of Theorem 1.1 is also proved.

Proposition 3.12. *Under the previous assumptions, the property* ***P)*** *holds for* (f_0,I_0,z_0).

We will prove this proposition in the following four cases:
- the component M_0 is a plane and $\rho_s(I,z_0)$ is reduced to 0;
- the component M_0 is neither a sphere nor a plane and $\rho_s(I,z_0)$ is reduced to 0;
- the component M_0 is a plane and $\rho_s(I,z_0)$ is reduced to a non-zero integer k;
- the component M_0 is neither a sphere nor a plane and $\rho_s(I,z_0)$ is reduced to an non-zero integer k.

We will use some results that will be deduced in the first two cases to obtain the last two cases.

The case where M_0 is a plane and $\rho_s(I,z_0)$ is reduced to 0

In this case, I_0 is a maximal identity isotopy on the plane M_0 that fixes only one point z_0 and satisfies $\rho_s(I_0,z_0)=\{0\}$. The result of Proposition 3.12 is just a corollary of

Theorem 3.4 and the following lemma:

Lemma 3.13. *Under the previous assumptions, there exists a periodic orbit of f in the annulus $M_0 \setminus \{z_0\}$.*

Proof. Of course, we can assume that z_0 is not accumulated by periodic orbits. As in Remark 3.3, one knows that z_0 is an indifferent fixed point with rotation number $\rho(I, z_0) = 0$.

Let $\mathcal{F}$ be a transverse foliation of I_0. One knows that $\mathcal{F}$ has a unique singularity z_0 and an end ∞. By the assumption in Remark 3.2, z_0 is a sink of $\mathcal{F}$. Since f_0 is area preserving and the total area of M_0 is finite, ∞ is a source of $\mathcal{F}$ and all the leaves of $\mathcal{F}$ are lines from ∞ to z_0. Let $\pi : \mathbb{R} \times (0,1) \to M_0 \setminus \{z_0\}$ be the universal cover such that the leaves of the lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}$ are the vertical lines oriented upward. Let $\widetilde{f}$ be the lift of f_0 associated to I_0, and $p_1 : \mathbb{R} \times (0,1) \to \mathbb{R}$ be the projection onto the first factor. Then we know that

$$p_1(\widetilde{f}(\widetilde{z}) - \widetilde{z}) > 0, \text{ for all } \widetilde{z} \in \mathbb{R} \times (0,1).$$

Let V be a small Jordan domain in the annulus $M_0 \setminus \{z_0\}$ such that $f(V) \cap V = \emptyset$. Let $\widetilde{V}$ be one of the connected components of $\pi^{-1}(V)$. By choosing V small enough, one can suppose that

$$|p_1(\widetilde{z}) - p_1(\widetilde{z}')| < \frac{1}{2} \text{ for all } \widetilde{z}, \widetilde{z}' \in \widetilde{V}.$$

Then, for every $z \in V$ and $\widetilde{z} \in \pi^{-1}\{z\}$, we know that

$$\frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n} \geq \frac{\left(\sum_{k=1}^{n} \chi_V(f^k(z))\right) - \frac{1}{2}}{n}.$$

We define $U = \cup_{k\in\mathbb{Z}} f^k(V)$. By Poincaré Recurrence Theorem, almost all points in U are recurrent. By Birkhoff-Khinchen Theorem, for almost all $z \in U$, and every $\widetilde{z} \in \pi^{-1}\{z\}$, both of the two limits

$$\lim_{n\to\infty} \frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n} \text{ and } \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\chi_V(f^k(z))}{n}$$

exist, and there exists a non negative measurable function φ on U that satisfies $\varphi \circ f = \varphi$ and

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{\chi_V(f^k(z))}{n} = \varphi(z) \text{ for almost all } z \in U.$$

Moreover, by Lebesgue's dominated convergence theorem,

$$\int_U \varphi = \int \chi_V = \mathrm{Area}(V) > 0.$$

Therefore, there exist a recurrent point $z \in V$ and $\widetilde{z} \in \pi^{-1}\{z\}$ such that the limit

$$\lim_{n\to\infty} \frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n}$$

exists and is positive. So, the rotation number of z is positive. We denote it by ρ.

On the other hand, let K_0 be a small enough continuum at z_0 whose rotation number is 0. We denote by $(M_0 \setminus K_0) \sqcup \mathbb{T}^1$ the prime-ends compactification at the end K_0, which is an annulus. We can extend f to $\mathbb{T}^1$ and know that the rotation number on $\mathbb{T}^1$ is 0. Then, there exists a fixed point on $\mathbb{T}^1$ whose rotation number is 0.

By the remark that follows Proposition 2.25, there exists a q-periodic orbit of rotation number p/q in the annulus $(M_0 \setminus K_0)$, for all irreducible $p/q \in (0, \rho)$. □

The case where M_0 is neither a sphere nor a plane and $\rho_s(I, z_0)$ is reduced to 0

Recall that f_0 is an area preserving homeomorphism of M_0, that z_0 is an isolated fixed point of f_0 satisfying $i(f_0, z_0) = 1$, that I_0 is a maximal identity isotopy that fixes only one point z_0 and satisfies $\rho_s(I_0, z_0) = \{0\}$.

As in Section 2.13, let $\pi : \widetilde{M} \to M_0 \setminus \{z_0\}$ be the annulus covering projection, $\widetilde{I}$ be the natural lift of I_0 to $\widetilde{M} \cup \{\star\}$, $\widetilde{f}$ be the lift of f_0 to $\widetilde{M} \cup \{\star\}$ associated to I_0. Then $\widetilde{I}$ is a maximal identity isotopy and $\mathrm{Fix}(\widetilde{I})$ is reduced to $\star$. For all irreducible $p/q \in \mathbb{Q}$, if O is a periodic orbit of type (p, q) associated to $\widetilde{I}$ at $\star$, then $\pi(O)$ is a periodic orbit of type (p, q) associated to I_0 at z_0. So, if the property **P)** holds for $(\widetilde{f}, \widetilde{I}, \star)$, then it holds for (f_0, I_0, z_0). The result of Proposition 3.12 is a corollary of Theorem 3.4 and the following Proposition 3.14, which is the most difficult part of this Chapter.

Proposition 3.14. *There exists a periodic orbit of $\widetilde{f}$ besides $\star$.*

The idea of the proof of the proposition is the following: we will first consider several simple situations such that there exists a periodic orbit of $\widetilde{f}$ besides $\star$, then we suppose that we are not in these situations and follow the idea of Le Calvez (see Section 11 of [LC05]) to get a contradiction.

Let us begin with some necessary assumptions and lemmas. Of course, we can suppose that $\star$ is not accumulated by periodic orbits of $\widetilde{f}$. As in Remark 3.3, $\star$ is an indifferent fixed point of $\widetilde{f}$ and the rotation number $\rho(\widetilde{I}, \star)$ is equal to 0. Let $\mathcal{F}$ be a transverse foliation of I_0, and $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$. By the assumption in Remark 3.2, z_0 is a sink of $\mathcal{F}$, and $\star$ is a sink of $\widetilde{\mathcal{F}}$. Denote by W the attracting basin of z_0 for $\mathcal{F}$, and by $\widetilde{W}$ the attracting basin of $\star$ for $\widetilde{\mathcal{F}}$. Write $\dot{W} = W \setminus \{z_0\}$ and $\dot{\widetilde{W}} = \widetilde{W} \setminus \{\star\}$. Recall that $\pi|_{\dot{\widetilde{W}}}$ is a homeomorphism between $\dot{\widetilde{W}}$ and $\dot{W}$ and can be extended continuously to a homeomorphism between $\widetilde{W}$ and W. The area on M_0 induces an area on $\widetilde{M}$. So $\widetilde{f}$ is area preserving, and the area of $\widetilde{W}$ is finite.

Lemma 3.15. *Under the previous assumptions, if there exists an invariant continuum $K \subset \widetilde{W}$ with positive area, then there exists a periodic orbit besides $\star$.*

Proof. The proof is similar to the proof of Lemma 3.13 except some small modifications when we try to find a recurrent point with positive rotation number. We will give a more precise description.

Since $\widetilde{W}$ is different from $\widetilde{M} \cup \{\star\}$, we can not get a lift of $\widetilde{f}$ as in the proof of Lemma 3.13. Instead, we will get a similar one by the following procedure. Let $\pi' : \mathbb{R}^2 \to \dot{\widetilde{W}}$ be a universal cover which sends the vertical lines upwards to the leaves of $\widetilde{F}|_{\dot{\widetilde{W}}}$. Since K is an invariant subset of $\widetilde{W}$, we can lift $\widetilde{f}|_{K \setminus \{\star\}}$ to a homeomorphism $\widehat{f}$ of $\pi'^{-1}(K \setminus \{\star\})$ such that

$$p_1(\widehat{f}(\widehat{z}) - \widehat{z}) > 0, \text{ for all } \widehat{z} \in \pi'^{-1}(K \setminus \{\star\}),$$

where p_1 is the projection onto the first factor.

Also, we should replace the small Jordan domain V in the proof of Lemma 3.13 with $V \cap K$ by choosing suitable V such that the area of $V \cap K$ is positive, that $f(V) \cap V = \emptyset$, and that for every component $\widehat{V}$ of $\pi'^{-1}(V)$, one has

$$|p_1(\widehat{z}) - p_1(\widehat{z}')| < 1/2 \quad \text{for all } \widehat{z}, \widehat{z}' \in \widehat{V}.$$

We can always find such a set because the area of K is positive. $\square$

Lemma 3.16. *Under the previous assumptions, if there exists an invariant continuum $K \subset \widetilde{W}$ containing $\star$ such that $\rho(\widetilde{I}, K) \neq 0$, then there exists a periodic orbit in $\widetilde{M}$.*

Proof. Recall that $\pi|_{\dot{\widetilde{W}}}$ is a homeomorphism between $\dot{\widetilde{W}}$ and $\dot{W}$. So, $\widetilde{W}$ is a proper subset of $\widetilde{M} \cup \{\star\}$, and the boundary of $\widetilde{W}$ is the union of some proper leaves. By Remark 2.22, one knows that $\widetilde{f}$ can be blown-up at ∞ and the blow-up rotation number $\rho(\widetilde{I}, \infty)$ is equal to 0.

We consider the prime-ends compactification of $\widetilde{M} \setminus K$ at the end K, and extend $\widetilde{f}$ continuously to a homeomorphism of $(\widetilde{M} \setminus K) \sqcup S^1$. We get a homeomorphism g of the closed annulus $S_\infty \sqcup (\widetilde{M} \setminus K) \sqcup S^1$ that coincides with $\widetilde{f}$ on $\widetilde{M} \setminus K$, where S_∞ is the circle we added when blowing-up $\widetilde{f}$ at ∞.

Moreover, g satisfies the intersection property and has different rotation numbers at each boundary, then by Proposition 2.24, there exists a periodic orbit in $\widetilde{M} \setminus K$, which is also a periodic orbit of $\widetilde{f}$. □

Lemma 3.17. *Suppose that there exists a closed disk $D \subset \widetilde{W}$ containing $\star$ as an interior point such that the connected component of $\bigcap_{k\in\mathbb{Z}} \widetilde{f}^{-k}(D)$ containing $\star$ is contained in the interior of D. Then $\widetilde{f}$ has another periodic orbit besides $\star$.*

Proof. We will proof this lemma by contradiction. Suppose that $\widetilde{f}$ does not have any other periodic orbit. Let K be the connected component of $\bigcap_{k\in\mathbb{Z}} \widetilde{f}^{-k}(D)$ containing $\star$. We identify K as a point $\{K\}$, and still denote by $\widetilde{f}$ the reduced homeomorphism. The fixed point $\{K\}$ is a non-accumulated saddle-point of $\widetilde{f}$ with index $i(\widetilde{f}, \{K\}) = i(\widetilde{f}, K) = i(\widetilde{f}, 0) = 1$. By Proposition 2.23, $\widetilde{f}$ can be blown-up at $\{K\}$ and $\rho(\widetilde{f}, \{K\})$ is different from $0 \in \mathbb{R}/\mathbb{Z}$. So, $\rho(\widetilde{I}, K)$ is different from 0. By the previous lemma, $\widetilde{f}$ has another periodic orbit besides $\star$, which is a contradiction. □

Now we begin the proof of Proposition 3.14.

Proof of Proposition 3.14. We will prove this proposition by contradiction. Suppose that there does not exist any other periodic orbit except $\star$. Let $(D_n)_{n\in N}$ be an increasing sequence of closed disks containing $\star$ as an interior point such that D_n is contained in the interior of D_{n+1} for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} D_n = \widetilde{W}$. Let K_n be the connected component of $\cap_{k\in\mathbb{Z}} \widetilde{f}^{-k}(D_n)$ containing $\star$. By Lemma 3.16, we know that $\rho(\widetilde{I}, K_n)$ is equal to 0 for every $n \in \mathbb{N}$. By Lemma 3.17, each K_n intersects the boundary of D_n. Let $K = \overline{\cup_{n\in\mathbb{N}} K_n} \subset \widetilde{M} \cup \{\star\}$. It is an invariant set of $\widetilde{f}$. The boundary of $\widetilde{W}$ is the union of proper leaves, so for every point in $\partial\widetilde{W}$, either its image or its pre-image by $\widetilde{f}$ will leave $\widetilde{W}$. Therefore, K can not touch the boundary of $\widetilde{W}$, and is included in $\widetilde{W}$. But each K_n intersects the boundary of D_n, so K intersects every neighborhood of ∞.

Lemma 3.18. *There does not exist any connected component of $\widetilde{M} \setminus K$ that is included in $\widetilde{W}$.*

Proof. We will give a proof by contradiction. Suppose that there exists a component $\widetilde{U}$ of $\widetilde{M} \setminus K$ such that $\widetilde{U} \subset \widetilde{W}$. Then $\partial\widetilde{U}$ is a subset of K, and so is $\partial(\widetilde{f}^n(\widetilde{U}))$ for every $n \in \mathbb{Z}$. So, $\widetilde{f}^n(\widetilde{U})$ is a component of $\widetilde{M} \setminus K$ for every $n \in \mathbb{Z}$. Recall that the area of $\widetilde{W}$ is finite, one deduces that the area of $\widetilde{U}$ is finite. So, for every $n \in \mathbb{Z}$, the area of $\widetilde{f}^n(\widetilde{U})$ is finite. Recall that K can not touch the boundary of $\widetilde{W}$, one knows that a component of $\widetilde{M} \setminus K$ that is not included in $\widetilde{W}$ contains a proper leaf in $\partial\widetilde{W}$ and hence has infinite area. So, $\widetilde{f}^n(\widetilde{U})$ is included in $\widetilde{W}$ for every $n \in \mathbb{Z}$, and hence $\cup_{n\in\mathbb{Z}} \widetilde{f}^n(\widetilde{U})$ is included in $\widetilde{W}$. Therefore, there exists $q \in \mathbb{N}$ such that $\widetilde{f}^q(\widetilde{U}) = \widetilde{U}$. Moreover, $\star$ is not an interior point of $\widetilde{U}$, and $\widetilde{U}$ is homeomorphic to a disk. Then, one deduces by the Brouwer plane translation theorem (see Section 2.4) that $\widetilde{f}^q$ has a fixed point in $\widetilde{U}$. Therefore, $\widetilde{f}$ has a periodic point different from $\star$. We get a contradiction. □

Let $\pi' : \widehat{M} \to \widetilde{M}$ be the universal cover, and T be a generator of the group of covering automorphisms. Let $\widehat{I} = (\widehat{f}_t)_{t\in[0,1]}$ be the natural lift of $\widetilde{I}$, and $\widehat{\mathcal{F}}$ be the lift of $\widetilde{\mathcal{F}}$. Write $\widehat{f} = \widehat{f}_1$. It is the lift of $\widetilde{f}$ associated to $\widetilde{I}$. Write $\widehat{K} = \pi'^{-1}(K \setminus \{\star\})$, and $\widehat{W} = \pi'^{-1}(\dot{\widetilde{W}})$.

Because K is connected and is adherent to both ends of $\widetilde{M}$, each connected component of $\widetilde{M}\setminus K$ is simply connected. So, if $\widetilde{U}$ is one of the connected components of $\widetilde{M}\setminus K$, and if $\widehat{U}$ is one of the components of $\pi'^{-1}(\widetilde{U})$, then $\widehat{U}$ does not intersect $T(\widehat{U})$. Therefore, $\widehat{M}\setminus\widehat{K}$ is not connected and has infinitely many components. By Lemma 3.18, each component of $\widehat{M} \setminus \widehat{K}$ contains a proper leaf in $\partial\widehat{W}$, and hence a disk bounded by this leaf. As in the following picture, this disk contains the image or the pre-image of this proper leaf by $\widehat{f}$.

Figure 3.1: Each component $\widehat{U}$ of $\widehat{M} \setminus \widehat{K}$ is invariant by $\widehat{f}$

So, every component of $\widehat{M} \setminus \widehat{K}$ is invariant by $\widehat{f}$.

Lemma 3.19. *Each leaf in* $\widetilde{W}$ *is an arc from* ∞ *to* $\star$.

Proof. Recall that the area of $\widetilde{W}$ is finite. So, there exist a leaf included in $\partial\widetilde{W}$ such that $\widetilde{W}$ is to its right and a leaf included in $\partial\widetilde{W}$ such that $\widetilde{W}$ is to its left. (Otherwise, $\widetilde{W}$ contains the positive or negative orbit of a wandering open set $\widetilde{W} \setminus \widetilde{f}(\widetilde{W})$ or $\widetilde{W} \setminus \widetilde{f}^{-1}(\widetilde{W})$ respectively.) Therefore, the following two situations can not occur, and each leaf in $\widetilde{W}$ is

an arc from ∞ to $\star$. □

Every leaf $\widehat{\Phi} \subset \widehat{W}$ divides $\widehat{M}$ into two part. We denote by $R(\widehat{\Phi})$ the component of $\widehat{M} \setminus \widehat{\Phi}$ to the right of $\widehat{\Phi}$ and by $L(\widehat{\Phi})$ the component to the left.

Lemma 3.20. *There does not exist any leaf* $\widehat{\Phi} \subset \widehat{W}$ *such that* $\widehat{\Phi} \subset \widehat{K}$.

Proof. We can prove this lemma by contradiction. Suppose that $\widehat{\Phi} \subset \widehat{K}$. Then a component of $\widehat{M} \setminus \widehat{K}$ is either to the left or to the right of $\widehat{\Phi}$. Moreover, if it is to the right (resp. left) of $\widehat{\Phi}$, it is to the right (resp. left) of $\widehat{f}(\widehat{\Phi})$. Therefore, $R(\widehat{\Phi}) \cap L(\widehat{f}(\widehat{\Phi}))$ is included in $\widehat{K}$, and so the interior of $\widehat{K}$ is not empty. We deduce that K is an invariant set of $\widetilde{f}$ with non-empty interior and finite area. By Lemma 3.15, there exists a periodic orbit of $\widetilde{f}$ in $\widetilde{M}$, which is a contradiction. □

Lemma 3.21. *Let $\widehat{\Phi}$ be a leaf in $\widehat{W}$, $t \mapsto \widehat{\Phi}(t)$ be an oriented parametrization of $\widehat{\Phi}$, and $\widehat{U}$ be a component of $\widehat{M} \setminus \widehat{K}$. If $\widehat{\Phi}$ intersects $\widehat{U}$, then both the area of $L(\widehat{\Phi}) \cap \widehat{U}$ and the area of $R(\widehat{\Phi}) \cap \widehat{U}$ are infinite, and there exists t_0 such that $\widehat{\Phi}(t) \in \widehat{U}$ for all $t \leq t_0$.*

Proof. We will first give a proof of the first statement by contradiction. We suppose that the area of $L(\widehat{\Phi}) \cap \widehat{U}$ is finite, the other case can be treated similarly. Then, $L(\widehat{\Phi}) \cap R(\widehat{f}^{-1}(\widehat{\Phi})) \cap \widehat{U}$ is a wandering open set whose negative orbit is contained in $L(\widehat{\Phi}) \cap \widehat{U}$. It contradicts the fact that $\widehat{f}$ is area preserving.

Let us prove the second statement. We know that both the area of $L(\widehat{\Phi}) \cap \widehat{U}$ and the area of $R(\widehat{\Phi}) \cap \widehat{U}$ are infinite. Since $\pi'|_{\widehat{U}}$ is injective, both the area of $\pi'(L(\widehat{\Phi}) \cap \widehat{U})$ and the area of $\pi'(R(\widehat{\Phi}) \cap \widehat{U})$ are infinite. The area of $\widetilde{W}$ is finite, so both $\pi'(L(\widehat{\Phi}) \cap \widehat{U})$ and $\pi'(R(\widehat{\Phi}) \cap \widehat{U})$ intersect $\widetilde{M} \setminus \widetilde{W}$, and hence both $L(\widehat{\Phi}) \cap \widehat{U}$ and $R(\widehat{\Phi}) \cap \widehat{U}$ intersect $\widehat{M} \setminus \widehat{W}$. Therefore, there exists a proper leaf $\widehat{\Phi}_1$ in $L(\widehat{\Phi}) \cap \widehat{U}$ and a proper leaf $\widehat{\Phi}_2$ in $R(\widehat{\Phi}) \cap \widehat{U}$. Fix a parametrization $t \mapsto \widehat{\Phi}_1(t)$ of $\widehat{\Phi}_1$ and a parametrization $t \mapsto \widehat{\Phi}_2(t)$ of $\widehat{\Phi}_2$, and draw a path γ in $\widehat{U}$ from a point of $\widehat{\Phi}_1$ to a point of $\widehat{\Phi}_2$. Let $s_1 = \inf\{t : \widehat{\Phi}_1(t) \in \gamma\}$, $s_2 = \sup\{t : \widehat{\Phi}_2(t) \in \gamma\}$, and γ' be the sub-path of γ connecting $\widehat{\Phi}_1(s_1)$ to $\widehat{\Phi}_2(s_2)$. Then, as in the following picture, $\Gamma = \widehat{\Phi}|_{(-\infty,s_1]}\gamma'\widehat{\Phi}_2|_{[s_2,\infty)}$ is an oriented proper arc and satisfies

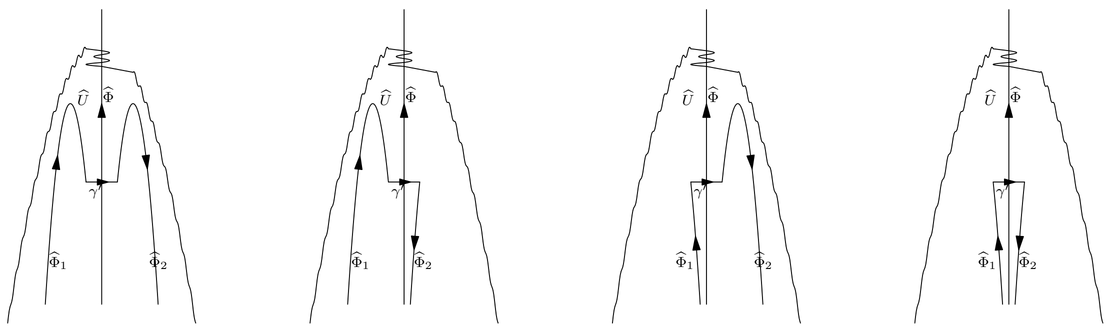


Figure 3.2: Four possible cases in the proof of Lemma 3.21

$R(\Gamma) \subset \widehat{U}$. We know that $\widehat{\Phi}$ intersects γ'. Let t_0 be a lower bound of the set $\{t : \widehat{\Phi}(t) \in \gamma'\}$. We know that $\widehat{\Phi}|_{(-\infty,t_0]} \subset \widehat{U}$. □

Let $\delta : \mathbb{T}^1 \to \widetilde{W}$ be an embedding that intersects $\widetilde{\mathcal{F}}$ transversely, and $\widehat{\delta} : \mathbb{R} \to \widehat{W}$ be the lift of δ. Then $\widehat{\delta}$ intersects every leaf in $\widehat{W}$, and intersects each leaf at only one point. Moreover, if $\widehat{\delta}$ intersects $\widehat{\Phi}$ and $\widehat{\Phi}'$ at $\widehat{\delta}(t)$ and $\widehat{\delta}(t')$ respectively, and if $t < t'$, then $\widehat{\Phi}$ is to the left of $\widehat{\Phi}'$, and $\widehat{\Phi}'$ is to the right of $\widehat{\Phi}$. We define a map h from $\mathbb{R}$ to the space of leaves of $\widehat{\mathcal{F}}$ in $\widehat{W}$ by $h(t) = \widehat{\Phi}$ if $\widehat{\delta}(t) \in \widehat{\Phi}$.

Lemma 3.22. *The set of points $t \in \mathbb{R}$ such that $h(t) \cap \widehat{U} \neq \emptyset$ is open for each component $\widehat{U}$ of $\widehat{W} \setminus \widehat{K}$.*

Proof. We fix a component $\widehat{U}$ of $\widehat{M} \setminus \widehat{K}$, and will first prove that the set $\{t : h(t) \cap \widehat{U} \neq \emptyset\}$ is open. Given a real number t such that $h(t)$ intersects $\widehat{U}$ and $z \in h(t) \cap \widehat{U}$, there is a trivialization neighborhood V of z such that $V \subset (\widehat{U} \cap \widehat{W})$. Moreover, $h^{-1}(V)$ is an open interval containing t. So, the set $\{t : h(t) \cap \widehat{U} \neq \emptyset\}$ is open. $\square$

By Lemma 3.20, each leaf of $\widehat{\mathcal{F}}$ in $\widehat{W}$ intersects at least a component of $\widehat{M} \setminus \widehat{K}$. By lemma 3.21, each leaf of $\widehat{\mathcal{F}}$ in $\widehat{W}$ intersects at most one component of $\widehat{M} \setminus \widehat{K}$. So, each leaf of $\widehat{\mathcal{F}}$ in $\widehat{W}$ intersects exactly one component of $\widehat{M} \setminus \widehat{K}$. Since $\widehat{M} \setminus \widehat{K}$ has countable components,

$$\mathbb{R} = \cup_{\widehat{U}}\{t : h(t) \cap \widehat{U} \neq \emptyset\}$$

is a disjoint union of countable many open sets. This is impossible. $\square$

The case where M_0 is a plane and $\rho_s(I, z_0)$ is reduced to a non-zero integer k

Recall that f_0 is an area preserving homeomorphism of M_0, that z_0 is an isolated fixed point of f_0 satisfying $i(f_0, z_0) = 1$, and that I_0 is a maximal identity isotopy that fixes only one point z_0 and satisfies $\rho_s(I_0, z_0) = \{k\}$. In this case, one can easily deduce that the result of Proposition 3.12 is just a corollary of the result in the previous two cases. We will give a brief explanation. Let J be the identity isotopy of the identity map on M_0 fixing z_0 and satisfying $\rho_s(J, z_0) = 1$. Write $I'_0 = J^{-k}I_0$. It is an identity isotopy of f_0 that satisfies $\rho_s(I'_0, z_0) = \{0\}$. By the result of Proposition 3.12 in the two cases we have already proved, the property **P)** holds for (f_0, I'_0, z_0). A periodic orbit in M_0 of type (p, q) associated to I'_0 at z_0 is a periodic orbit of type $(kq + p, q)$ associated to I_0 at z_0. So, the property **P)** holds for (f_0, I_0, z_0).

The case where M_0 is neither a sphere nor a plane and $\rho_s(I, z_0)$ is reduced to a non-zero integer k

Recall that f_0 is an area preserving homeomorphism of M_0, that z_0 is an isolated fixed point of f_0 satisfying $i(f_0, z_0) = 1$, that I_0 is a maximal identity isotopy that fixes only one point z_0 and satisfies $\rho_s(I_0, z_0) = \{k\}$.

As in Section 2.13, let $\pi : \widetilde{M} \to M_0 \setminus \{z_0\}$ be the annulus covering projection, $\widetilde{I}$ be the natural lift of I_0 to $\widetilde{M} \cup \{\star\}$, and $\widetilde{f}$ be the lift of f_0 associated to I_0 to $\widetilde{M} \cup \{\star\}$. Then $\widetilde{I}$ is a maximal identity isotopy and $\mathrm{Fix}(\widetilde{I})$ is reduced to $\star$. As before, if the property **P)** holds for $(\widetilde{f}, \widetilde{I}, \star)$, it holds for (f_0, I_0, z_0).

Let $\mathcal{F}$ be a transverse foliation of I_0, and $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$. Since $\rho_s(I_0, z_0)$ is reduced to a non-zero integer, by the assertion v) of Proposition 2.20, z_0 is a sink or a source of $\mathcal{F}$ and $\star$ is a sink or a source of $\widetilde{\mathcal{F}}$. Let W be the attracting or repelling basin of z_0 for $\mathcal{F}$, and $\widetilde{W}$ be the attracting or repelling basin of $\star$ for $\widetilde{\mathcal{F}}$. Recall that $\pi|_{\widetilde{W}\setminus\{\star\}}$ is a homeomorphism be tween $\widetilde{W} \setminus \{\star\}$ and $W \setminus \{z_0\}$. So, $\widetilde{W}$ is a strict subset of $\widetilde{M} \cup \{\star\}$, and its boundary is the union of some proper leaves. By Remark 2.22, one knows that $\widetilde{f}$ can be blown-up at ∞ and $\rho(\widetilde{I}, \infty)$ is equal to 0.

Let J be the identity isotopy of the identity map of $\widetilde{M} \cup \{\star\}$ fixing $\star$ and satisfying $\rho_s(J, \star) = 1$. Write $\widetilde{I}' = J^{-k}\widetilde{I}$. We know that $\rho_s(\widetilde{I}', \star)$ is reduced to 0, and that the blow-up rotation number $\rho(\widetilde{I}', \infty)$ is equal to k. One deduces by the assertion ii) of Proposition 2.20 that there exists a neighborhood of ∞ that does not contain any contractible fixed points of $\widetilde{f}|_{\widetilde{M}}$ associated to $\widetilde{I}'|_{\widetilde{M}}$. Let $(Y, \widetilde{I}_Y)$ be a maximal extension of $(\{\star\}, \widetilde{I}')$ (see Section 2.6). One knows that Y is a closed subset of the union of $\{\star\}$ and the set of contractible fixed points of $\widetilde{f}|_{\widetilde{M}}$ associated to $\widetilde{I}'|_{\widetilde{M}}$. So, there is a neighborhood of ∞ that does not intersect Y, and hence Y is a compact subset in $\widetilde{M} \cup \{\star\}$. One knows also that $\rho_s(\widetilde{I}_Y, \star)$ is reduced to 0, and that the blow-up rotation number $\rho(\widetilde{I}_Y, \infty)$ is equal to k. As in the previous subsection, in order to prove the result of Proposition 3.12, we only need to prove that the property **P)** holds for $(\widetilde{f}, \widetilde{I}_Y, \star)$, which is the aim of this subsection.

Proposition 3.23. *Under the previous assumptions, the property* ***P)*** *holds for* $(\widetilde{f}, \widetilde{I}_Y, \star)$.

Proof. To get this result, one has to consider two cases: Y is reduced to a single point $\star$ or it contains at least two points. In the first case, the proposition is a corollary of Lemma 3.7. Now, we will prove the proposition in the second case.

Suppose that Y contains at least two points and write $Y_0 = Y \setminus \{\star\}$. Let $\widetilde{M}_0$ be the connected component of $\widetilde{M} \cup \{\star\} \setminus Y_0$ containing $\star$. Recall that Y is a compact subset of $\widetilde{M} \cup \{\star\}$. So, one has to consider the following two cases:

- $\widetilde{M}_0$ is a bounded plane,
- $\widetilde{M}_0$ is neither a sphere nor a plane.

In the first case, the area of $\widetilde{M}_0$ is finite, and the problem is reduced to the case in the first part of Section 3.1.2; while in the second case, we will prove the result like in the second part of Section 3.1.2.

Now, we suppose that $\widetilde{M}_0$ is neither a sphere nor a plane. Let $\pi'' : \check{M} \to \widetilde{M}_0$ be an annulus covering map, $\check{I}$ be the natural lift of $\widetilde{I}_Y|_{\widetilde{M}_0}$ to $\check{M} \cup \{\check{\star}\}$, and $\check{f}$ be the lift of $\widetilde{f}|_{\widetilde{M}_0}$ to $\check{M} \cup \{\check{\star}\}$ associated to $\widetilde{I}_Y|_{\widetilde{M}_0}$. As before, if the Property **P)** holds for $(\check{f}, \check{I}, \check{\star})$, then it holds also for $(\widetilde{f}, \widetilde{I}_Y, \star)$. So, the proposition is a corollary of Theorem 3.4 and the following Lemma 3.24. $\square$

Lemma 3.24. *There exists a periodic orbit of* $\check{f}$ *besides* $\check{\star}$.

Proof. The proof is similar to the proof of Proposition 3.14.

Let $\widetilde{\mathcal{F}}_Y$ be a transverse foliation of $\widetilde{I}_Y$, and $\check{\mathcal{F}}$ be the lift of $\mathcal{F}_Y|_{\widetilde{M}_0}$ to $\check{M} \cup \{\check{\star}\}$. Recall the assumption in Remark 3.2, one knows that $\star$ is a sink of $\widetilde{\mathcal{F}}_Y$ and that $\check{\star}$ is a sink of $\check{\mathcal{F}}$. Let $\widetilde{W}^*$ be the attracting basin of $\star$ for $\widetilde{\mathcal{F}}_Y$ and $\check{W}$ be the attracting basin of $\check{\star}$ for $\check{\mathcal{F}}$. Recall that $\pi''|_{\check{W} \setminus \{\check{\star}\}}$ is a homeomorphism between $\check{W} \setminus \{\check{\star}\}$ and $\widetilde{W}^* \setminus \{\star\}$.

When $k \geq 1$, one deduces by Proposition 2.20 that the end ∞ is sink of $\widetilde{\mathcal{F}}_Y$. In this case $\widetilde{W}^*$ is a bounded subset of $\widetilde{M} \cup \{\star\}$, and hence the area of both $\widetilde{W}^*$ and $\check{W}$ are finite. We can repeat the proof of Proposition 3.14, and get the result.

Now, we suppose that $k \leq -1$. In this case, the end ∞ is a source of of $\widetilde{\mathcal{F}}_Y$.

Sublemma 3.25. *Each leaf in* $\check{W}$ *is an arc from infinite to* $\check{\star}$.

Proof. When the area of $\check{W}$ is finite, we deduces the result as in Lemma 3.19. Now suppose that the area of $\check{W}$ is infinite. We consider the compactification of $\widetilde{M} \cup \{\star\}$ by adding a point ∞ at infinite, the added point ∞ is a source of $\widetilde{F}_Y$ and is at the boundary of $\widetilde{W}^*$. So, there exists a leaf in $\widetilde{W}^*$ from the singularity ∞ to $\star$ whose lift in $\check{W}$ is a leaf from infinite to $\check{\star}$. Therefore, each leaf in $\check{W}$ is an arc from infinite to $\check{\star}$. $\square$

The difference between our case and the case of Proposition 3.14 is that the area of $\check{W}$ may be infinite. But we did not use this condition except in the proof of Lemma 3.19 and Lemma 3.20. We have proven Sublemma 3.25 corresponding to Lemma 3.19. We will prove that the area of K is finite, so the result of Lemma 3.20 is still valid.

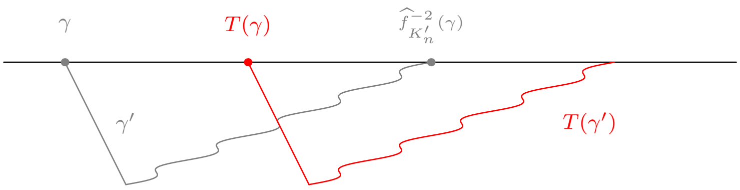
Formally, suppose that there does not exist any periodic orbits besides $\check{\star}$. Let $(D_n)_{n\in N}$ be an increasing sequence of closed disks containing $\check{\star}$ such that D_n is contained in the interior of D_{n+1} for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} D_n = \check{W}$. Let K_n be the connected component of $\cap_{k\in\mathbb{Z}} \check{f}^{-k}(D_n)$ containing $\check{\star}$ and $K = \overline{\cup_{n\in\mathbb{N}} K_n} \subset \check{M} \cup \{\star\}$. We will prove that the area of K is finite.

Let $K'_n = \pi''(K_n)$, and $K' = \overline{\cup_n K'_n} \subset \widetilde{M} \cup \{\star\} \cup S_\infty$, where S_∞ is the circle we added when blowing-up $\widetilde{f}$ at ∞. As before, we can deduce that $K \subset \breve{W}$. Recall that $\pi''|_{\breve{W}\setminus\{\breve{\star}\}}$ is a homeomorphism between $\breve{W} \setminus \{\breve{\star}\}$ and $\widetilde{W}^* \setminus \{\star\}$. Therefore, we know that $\pi''(K) \subset K'$, and that the area of K is not bigger than the area of K'. So, we only need to prove that the area of K' is finite.

We will prove it by contradiction. Suppose that the area of K' is infinite. One deduces that $K' \cap S_\infty \neq \emptyset$. As was proven in Section 3.1.2, one knows that $\rho(\breve{I}, K_n) = 0$, and so $\rho(\widetilde{I}_Y, K'_n) = 0$ for all $n \in \mathbb{N}$. Since $(Y, \widetilde{I}_Y)$ is a maximal extension of $(\{\star\}, J^{-k}\widetilde{I})$, one deduces that $\rho(J^{-k}\widetilde{I}, K'_n)$ is equal to 0 and that $\rho(\widetilde{I}, K'_n)$ is equal to k, for all $n \in \mathbb{N}$.

Since $K' \cap S_\infty$ is invariant by $\widetilde{f}$ and the blow-up rotation number $\rho(\widetilde{I}, \infty) = 0$, there exists a fixed point $\widetilde{z}_1 \in K' \cap S_\infty$, and the rotation number of $\widetilde{z}_1$ (associated to $\widetilde{I}$) in the annulus $\widetilde{M} \cup S_\infty$ is 0.

Let $\pi' : \widehat{M} \to \widetilde{M} \cup S_\infty$ be the universal cover, T be a generator of the group of covering automorphism, and $\widehat{f}$ the lift of $\widetilde{f}$ associated to $\widetilde{I}$. Fix one $\widehat{z}_1 \in \pi'^{-1}(\widetilde{z}_1)$. It is a fixed point of $\widehat{f}$. Let U be a small neighborhood of $\widehat{z}_1$ such that $T^n(U) \cap U = \emptyset$ for all $n \neq 0$. Let $V \subset U$ be a neighborhood of $\widehat{z}_1$ such that $\widehat{f}^2(V) \subset U$. Fix n large enough such that $K'_n \cap V \neq \emptyset$, and choose an arc γ in V connecting $\widehat{z}_1$ and an accessible point of K'_n such that $\gamma \cap K'_n$ has exactly one point. By choosing a sub-arc of $\gamma \cup \widehat{f}^{-2}(\gamma)$, we get a cross-cut γ'. On one hand, $T(\gamma') \cap \gamma' = \emptyset$ because $\gamma' \subset V$. On the other hand, we consider the prime-ends compactification of $\widetilde{M} \cup S_\infty \setminus K'_n$ at the end K'_n, and denote by $\widetilde{f}_{K'_n}$ the extension of $\widetilde{f}|_{\widetilde{M}\setminus K'_n}$. As was in Section 2.9, let $\pi'_{K'_n} : \pi'^{-1}(\widetilde{M} \cup S_\infty \setminus K'_n) \cup \mathbb{R} \to (\widetilde{M} \cup S_\infty \setminus K'_n) \cup S^1$ be the universal cover, and $\widehat{f}_{K'_n}$ the lift of $\widetilde{f}_{K'_n}$ whose restriction to $\pi'^{-1}(\widetilde{M} \cup S_\infty \setminus K'_n)$ is equal to $\widehat{f}$. Recall that $\rho(\widetilde{I}, K'_n)$ is equal to $k \leq -1$. So, the end-cut $\widehat{f}^{-2}_{K'_n}(\gamma) > T^{-2k-1}(\gamma) \geq T(\gamma)$,



which means $\gamma' \cap T(\gamma') \neq \emptyset$. We get a contradiction. $\square$

3.2 The case of diffeomorphisms

3.2.1 The index at a degenerate fixed point that is an extremum of a generating function

Let f be a diffeomorphism of $\mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^2$ function, we call g a *generating function* of f if $\partial^2_{12} g < 1$, and if

$$f(x, y) = (X, Y) \Leftrightarrow \begin{cases} X - x = & \partial_2 g(X, y), \\ Y - y = & -\partial_1 g(X, y). \end{cases} \tag{3.1}$$

Every $\mathcal{C}^2$ function $g : \mathbb{R}^2 \to \mathbb{R}$ satisfiying $\partial^2_{12} g \leq c < 1$ defines a diffeomorphism f of $\mathbb{R}^2$ by the previous equations. Moreover, the Jacobian matrix J_f of f is equal to

$$\frac{1}{1 - \partial^2_{12} g(X, y)} \begin{pmatrix} 1 & \partial^2_{22} g(X, y) \\ -\partial^2_{11} g(X, y) & -\partial^2_{11} g(X, y)\partial^2_{22} g(X, y) + (1 - \partial^2_{12} g(X, y))^2 \end{pmatrix}.$$

Since $\det J_f = 1$, the diffeomorphism f is orientation and area preserving. On the other hand, every orientation and area preserving diffeomorphism f of $\mathbb{R}^2$ satisfying $0 < \varepsilon \leq \partial_1(p_1 \circ f) \leq M < \infty$ can be generated by a generating function, where p_1 is the projection onto the first factor.

Moreover, we can naturally define an identity isotopy $I_0 = (f_t)_{t\in[0,1]}$ of f such that f_t is generated by tg. Precisely, the diffeomorphisms f_t are defined by the following equations:

$$f_t(x,y) = (X^t, Y^t) \Leftrightarrow \begin{cases} X^t - x = & t\partial_2 g(X^t, y), \\ Y^t - y = & -t\partial_1 g(X^t, y). \end{cases} \tag{3.2}$$

A point (x, y) is a fixed point of f if and only if it is a critical point of g. We say that a fixed point (x, y) of f is *degenerate* if 1 is an eigenvalue of $J_f(x, y)$. We will see later that a fixed point (x, y) of f is degenerate if and only if the Hessian matrix of g at (x, y) is degenerate.

We can also define a local generating function. Precisely, if (x, y) is a critical point of a $\mathcal{C}^2$ function g such that $\partial_{12}g(x, y) < 1$, then one can define an orientation and area preserving local diffeomorphism f at (x, y) by the equations (3.1). On the other side, if (x, y) is a fixed point of an orientation and area preserving diffeomorphism f such that $\partial_1(p_1 \circ f)(x, y) > 0$, where p_1 is the projection to the first factor, then one can find a $\mathcal{C}^2$ function g defined in a neighborhood of (x, y), that defines the germ of f at (x, y) by the equations (3.1). Moreover, in both cases, we can define a local isotopy of f at (x, y) by the equations (3.2), and will call it the *local isotopy induced by g*.

In this section, suppose that $f : (W, 0) \to (W', 0)$ is a local diffeomorphism at $0 \in \mathbb{R}^2$, and that g is a local generating function of f. We will prove the following Proposition 3.26, and deduce Corollary 1.3 as an immediate consequence of Theorem 1.1 and Proposition 3.26.

Proposition 3.26. *If 0 is an isolated critical point of g and a local extremum of g, and if the Hessian matrix of g at 0 is degenerate, then $i(f, 0)$ is equal to 1.*

Proof. The idea is to compute the indices of the local isotopies, so that we can deduce the Lefschetz index by Proposition 2.1.

We denote by $I_0 = (f_t)_{t\in[0,1]}$ the local isotopy induced by g. We have the following lemma

Lemma 3.27. *The blow-up rotation number $\rho(I_0, 0)$ is equal to 0.*

Proof. Since $\mathrm{Hess}(g)(0)$ is degenerate, one deduces that 0 is an eigenvalue of $\mathrm{Hess}(g)(0)$. Let v be an eigenvector of $\mathrm{Hess}(g)(0)$ corresponding to the eigenvalue 0. We will prove that v is a common eigenvector of $J_{f_t}(0)$ corresponding to the eigenvalue 1 for $t \in [0, 1]$, and hence the blow-up rotation number $\rho(I_0, 0)$ is equal to 0.

Write

$$\mathrm{Hess}(g)(0) = \begin{pmatrix} \varrho & \sigma \\ \sigma & \tau \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Then, one deduces that

$$\varrho\tau - \sigma^2 = 0, \quad \varrho a + \sigma b = 0, \quad \text{and } \sigma a + \tau b = 0.$$

By a direct computation, one knows that for every $t \in [0, 1]$,

$$J_{f_t}(0) = \frac{1}{1 - t\sigma} \begin{pmatrix} 1 & t\tau \\ -t\varrho & -t^2\varrho\tau + (1 - t\sigma)^2 \end{pmatrix} = \frac{1}{1 - t\sigma} \begin{pmatrix} 1 & t\tau \\ -t\varrho & 1 - 2t\sigma \end{pmatrix},$$

and then

$$J_{f_t}(0)v = \frac{1}{1-t\sigma}\begin{pmatrix} a+t\tau b \\ -t\varrho a + b - 2t\sigma b\end{pmatrix} = \frac{1}{1-t\sigma}\begin{pmatrix} a-t\sigma a \\ t\sigma b + b - 2t\sigma b\end{pmatrix} = \begin{pmatrix} a \\ b\end{pmatrix}.$$

□

Since f is area preserving, the rotation set at 0 is not empty. By the assertion vii) of Proposition 2.20, and the previous lemma, one can deduce that $\rho_s(I_0, 0)$ is reduced to 0, and that for all local isotopy I of f that is not equivalent to I_0, the rotation set $\rho_s(I, 0)$ is reduced to a non-zero integer.

Lemma 3.28. *If I is a local isotopy of f that is not equivalent to I_0, then $i(I, 0)$ is equal to 0.*

Proof. Let $\mathcal{F}$ be foliation locally transverse to I. Since $\rho_s(I, 0)$ is reduced to a non-zero integer, one can deduce by the assertion v) of Proposition 2.20 that 0 is either a sink or a source of $\mathcal{F}$. By Proposition 2.3, one deduces that $i(I, 0) = i(\mathcal{F}, 0) - 1 = 0$. □

In order to compute the index of I_0, we will construct an isotopy I' that is equivalent to I_0, and prove that $i(I', 0) = 0$.

We define $I' = (f'_t)_{t\in[0,1]}$ in a neighborhood of 0 by

$$f'_t(x, y) = \begin{cases} (x, y) + 2t(X - x, 0) & \text{for } 0 \le t \le 1/2, \\ (X, y) + (2t-1)(0, Y - y) & \text{for } 1/2 \le t \le 1, \end{cases}$$

where $(X, Y) = f(x, y)$.

Lemma 3.29. *The family $I' = (f'_t)_{t\in[0,1]}$ is a local isotopy of f.*

Proof. For every fixed $t \in [0, 1]$, We will prove that f'_t is a local diffeomorphism by computing the determinant of the Jacobian matrices, and then get the result.

Indeed, one knows

$$\partial_1 X = 1/(1 - \partial_{12} g) > 0.$$

Then for $t \in [0, 1/2]$,

$$\det J_{f'_t} = \det\begin{pmatrix} 1 + 2t(\partial_1 X - 1)) & 2t\partial_2 X \\ 0 & 1\end{pmatrix} = 2t\partial_1 X + (1 - 2t) > 0;$$

and for $t \in [1/2, 1]$,

$$\det J_{f'_t} = \det\begin{pmatrix} \partial_1 X & \partial_2 X \\ (2t-1)\partial_1 Y & (2-2t) + (2t-1)\partial_2 Y\end{pmatrix} = (2t-1)\det J_f + (2-2t)\partial_1 X > 0.$$

□

Lemma 3.30. *The blow-up rotation number $\rho(I', 0)$ is equal to 0, and hence I' is equivalent to I_0.*

Proof. As in the proof of Lemma 3.27, we will prove that an eigenvector of $\mathrm{Hess}(g)(0)$ corresponding to the eigenvalue 0 is a common eigenvector of $J_{f'_t}(0)$ corresponding to the eigenvalue 1 for $t \in [0,1]$, and hence deduce the lemma.

We keep the notations in the proof of Lemma 3.27, and recall that

$$\varrho\tau - \sigma^2 = 0, \quad \varrho a + \sigma b = 0, \quad \text{and } \sigma a + \tau b = 0.$$

For $t \in [0, 1/2]$,

$$J_{f'_t}(0) = \mathrm{Id} + 2t \begin{pmatrix} \partial_1 X(0,0) - 1 & \partial_2 X(0,0) \\ 0 & 0 \end{pmatrix} = \mathrm{Id} + \frac{2t}{1-\sigma} \begin{pmatrix} \sigma & \tau \\ 0 & 0 \end{pmatrix},$$

and

$$J_{f'_t}(0)v = v + \frac{2t}{1-\sigma} \begin{pmatrix} \sigma a + \tau b \\ 0 \end{pmatrix} = v.$$

For $t \in [1/2, 1]$,

$$J_{f'_t}(0) = J_f(0) - (2-2t) \begin{pmatrix} 0 & 0 \\ \partial_1 Y(0,0) & \partial_2 Y(0,0) - 1 \end{pmatrix} = J_f(0) - \frac{2-2t}{1-\sigma} \begin{pmatrix} 0 & 0 \\ -\varrho & -\sigma \end{pmatrix},$$

and

$$J_{f'_t}(0)v = J_f(0)v + \frac{2-2t}{1-\sigma} \begin{pmatrix} 0 \\ \varrho a + \sigma b \end{pmatrix} = v.$$

We have verified that v is a common eigenvector of $J_{f'_t}(0)$ corresponding to the eigenvalue 1 for $t \in [0,1]$. $\square$

To conclude, we will define a locally transverse foliation $\mathcal{F}_0$ of I' such that 0 is a sink or a source of $\mathcal{F}_0$, and then deduce by Proposition 2.3 that $i(I', 0) = i(\mathcal{F}_0, 0) - 1 = 0$. Indeed, let $\mathcal{F}_0$ be the foliation in a neighborhood of 0 whose leaves are the integral curves of the gradient vector field[1] of g. One knows that 0 is a sink of $\mathcal{F}_0$ if 0 is a local maximum of g, and is a source of $\mathcal{F}_0$ if 0 is a minimum of g. We can finish our proof by the following lemma.

Lemma 3.31. *The foliation $\mathcal{F}_0$ is locally transverse to I'.*

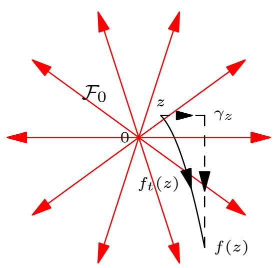


Figure 3.3: The dynamics and foliation generated by $g(x, y) = x^2 + y^2$

Proof. Let U be a sufficiently small Jordan domain containing 0 such that $\mathcal{F}_0$ is well defined on U, and $V \subset U$ be a sufficiently small neighborhood of 0 such that f'_t is well defined on V for $t \in [0,1]$, that f does not have any other fixed point in V except 0,

1. It means the vector field: $(x, y) \mapsto (\partial_1 g(x,y), \partial_2 g(x,y))$.

and that $\cup_{t\in[0,1]} f'_t(V) \subset U$. We will prove that for every $z = (x,y) \in V \setminus \{0\}$, the path $\gamma_z : t \mapsto f'_t(x,y)$ is positively transverse to $\mathcal{F}_0$, and then deduce the lemma.

Indeed, for $t \in [0,1/2]$,

$$\begin{aligned}
& \det \begin{pmatrix} 2(X-x) & \partial_1 g(f'_t(x,y)) \\ 0 & \partial_2 g(f'_t(x,y)) \end{pmatrix} \\
= \ & 2(X-x)\partial_2 g(f'_t(x,y)) \\
= \ & 2(X-x)\partial_2 g(2tX + (1-2t)x, y) \\
= \ & 2(X-x)[\partial_2 g(X,y) + (2t-1)(X-x)\partial^2_{12} g(\xi,y)] \\
= \ & 2(X-x)^2[1-(1-2t)\partial^2_{12} g(\xi,y)] \geq 0
\end{aligned}$$

where ξ is a real number between x and X, and the inequality is strict if $X \neq x$.

For $t \in [1/2,1]$,

$$\begin{aligned}
& \det \begin{pmatrix} 0 & \partial_1 g(f'_t(x,y)) \\ 2(Y-y) & \partial_2 g(f'_t(x,y)) \end{pmatrix} \\
= \ & -2(Y-y)\partial_1 g(f'_t(x,y)) \\
= \ & -2(Y-y)\partial_1 g(X, (2-2t)y + (2t-1)Y) \\
= \ & -2(Y-y)[\partial_1 g(X,y) + (2t-1)(Y-y)\partial^2_{12} g(X,\eta)] \\
= \ & 2(Y-y)^2[1-(2t-1)\partial^2_{12} g(X,\eta)] \geq 0
\end{aligned}$$

where η is a real number between y and Y, and the inequality is strict if $Y \neq y$.

Since $z = (x,y)$ is not a fixed point, either $X \neq x$ or $Y \neq y$. If both of the inequalities are satisfied, γ_z is positively transverse to $\mathcal{F}_0$; if $X \neq x$ and $Y = y$, $\gamma_z|_{t\in[0,\frac{1}{2}]}$ is positively transverse to $\mathcal{F}_0$, and $\gamma_z|_{t\in[\frac{1}{2},1]}$ is reduced to a point; if $X = x$ and $Y \neq y$, $\gamma_z|_{t\in[0,\frac{1}{2}]}$ is reduced to a point, and $\gamma_z|_{t\in[\frac{1}{2},1]}$ is positively transverse to $\mathcal{F}_0$. □

□

Remark 3.32. In the proof, we have indeed proven that $\mathcal{F}_0$ is locally transverse to any local isotopy of f that is equivalent to I_0.

3.2.2 Discrete symplectic actions and symplectically degenerate extrema

In this section, we will introduce symplectically degenerate extrema. More details can be found in [Maz13].

We say that a diffeomorphism $F : \mathbb{T}^2 \to \mathbb{T}^2$ is *Hamiltonian* if it is area preserving and if there exists a lift f satisfying

$$f(z+k) = f(z) + k \text{ for all } k \in \mathbb{Z}^2, \text{ and } \int_{\mathbb{T}^2} (f - \mathrm{Id})dxdy = 0.$$

Refering to [MS98], this definition coincides with the usual definition of a Hamiltonian diffeomorphism of a symplectic manifold. More precisely, we call a time-dependent vector field $(X_t)_{t\in\mathbb{R}}$ a *Hamiltonian vector field* if it is defined by the equation:

$$dH_t = \omega(X_t, \cdot),$$

where (M,ω) is a symplectic manifold and $H : \mathbb{R} \times M \to \mathbb{R}$ is a smooth function. The Hamiltonian vector field induces a *Hamiltonian flow* $(\varphi_t)_{t\in\mathbb{R}}$ on M, which is the solution of the following equation

$$\frac{\partial}{\partial t}\varphi_t(z) = X_t(\varphi_t(z)).$$

We say that a diffeomorphism F of M is a *Hamiltonian diffeomorphism* if it is the time-1 map of a Hamiltonian flow. So, for a Hamiltonian diffeomorphism, there exists a natural identity isotopy I which is defined by the Hamiltonian flow. We say that a fixed point of a Hamiltonian diffeomorphism is *contractible* if its trajectory along I is a loop homotopic to zero in M, and that a q-periodic point of a Hamiltonian diffeomorphism is *contractible* if it is a contractible fixed point of F^q.

Let $F : \mathbb{T}^2 \to \mathbb{T}^2$ be a Hamiltonian diffeomorphism. Then F is the time-1 map of a Hamiltonian flow, and we can factorize F by

$$F = F_{k-1} \circ \cdots \circ F_0,$$

where F_j is $\mathcal{C}^1$-close to the identity, for $j = 0, \cdots, k-1$. For every j, let f_j be the lift of F_j that is $\mathcal{C}^1$-close to the identity, and g_j be a generating function of f_j. We define the *discrete symplectic action*

$$g : \mathbb{R}^{2k} \to \mathbb{R}$$

by

$$g(z) := \sum_{j\in\mathbb{Z}_k} (< y_j, x_j - x_{j+1} > + g_j(x_{j+1}, y_j)),$$

where $z = (z_0, ..., z_{k-1})$ and $z_j = (x_j, y_j)$.

By a direct computation, we know that for every $j \in \mathbb{Z}_k$,

$$\frac{\partial}{\partial x_j} g(z) = y_j - y_{j-1} + \partial_1 g_{j-1}(x_j, y_{j-1}), \quad \text{and } \frac{\partial}{\partial y_j} g(z) = x_j - x_{j+1} + \partial_2 g_j(x_{j+1}, y_j).$$

So, $z \in \mathbb{R}^{2k}$ is a critical point of g if and only if $z_{j+1} = f_j(z_j)$ for every $j \in \mathbb{Z}_k$, and therefore if and only if $z_0 \in \mathbb{R}^2$ is a fixed point of $f = f_{k-1} \circ \cdots \circ f_0$.

In particular, each f_j commutes with the integer translation, and so g is invariant by the diagonal action of $\mathbb{Z}^2$ on $\mathbb{R}^{2k}$ and descends to a function

$$G : \mathbb{R}^{2k}/\mathbb{Z}^2 \to \mathbb{R}.$$

Moreover, $[z] \in \mathbb{R}^{2k}/\mathbb{Z}^2$ is a critical point of G if and only if $z_{j+1} = f_j(z_j)$ for every $j \in \mathbb{Z}_k$, and therefore if and only if $[z_0] \in \mathbb{T}^2$ is a contractible fixed point of F, where $F = F_{k-1} \circ \cdots \circ F_0$. In particular, critical points of G one-to-one correspond to contractible fixed points of F. Moreover, for any period $q \in \mathbb{N}$, contractible q-periodic points of F correspond to the equivalent classes in $\mathbb{R}^{2kq}/\mathbb{Z}^2$ of critical points of the discrete symplectic action $g^{\times q} : \mathbb{R}^{2kq}/\mathbb{Z}^2 \to \mathbb{R}$ defined by

$$g^{\times q}(z) := \sum_{j\in\mathbb{Z}_{kq}} (< y_j, x_j - x_{j+1} > + g_{(j \mod k)}(x_{j+1}, y_j)),$$

where $z = (z_0, ..., z_{kp-1})$ and $z_j = (x_j, y_j)$.

Moreover, if $[z_0] \in \mathbb{T}^2$ is a contractible fixed point of F, then by a suitable shift one can suppose that $[z_0]$ is fixed along the Hamiltonian flow, and hence the factors F_j fixes $[z_0]$ for $j = 0, \cdots, k-1$. So, z_0 is a fixed point of each f_j and a critical point of each g_j. We denote by $C_*(z_0^{\times kn})$ the graded group of relative homology

$$H_*(\{g^{\times n} < g^{\times n}(z_0^{\times kn})\} \cup \{z_0^{\times kn}\}, \{g^{\times n} < g^{\times n}(z_0^{\times kn})\}).$$

Then $C_j(z_0^{\times kn})$ are always trivial for $j < \mathrm{mor}(z_0^{\times kn})$ and $j > \mathrm{mor}(z_0^{\times kn})+\mathrm{nul}(z_0^{\times kn})$, where $\mathrm{mor}(z_0^{\times kn})$ is the dimension of negative eigenvector space of Hessian matrix of $g^{\times n}$ at $z_0^{\times kn}$, and $\mathrm{nul}(z_0^{\times kn})$ is the dimension of the kernel of Hessian matrix of $g^{\times n}$ at $z_0^{\times kn}$.

We say that z_0 is a *symplectically degenerate maximum* if z_0 is an isolated local maximum of the generating functions $g_0, \cdots, g_{k-1}$, and the local homology $C_{kn+1}(z_0^{\times kn})$ is non-trivial for infinitely many $n \in \mathbb{N}$.

Similarly, we denote by $C_*^+(z_0^{\times kn})$ the graded group of relative homology

$$H_*(\{g^{\times n} > g^{\times n}(z_0^{\times kn})\} \cup \{z_0^{\times kn}\}, \{g^{\times n} > g^{\times n}(z_0^{\times kn})\}),$$

and say that z_0 is a *symplectically degenerate minimum* if z_0 is an isolated local minimum of the generating functions $g_0, \cdots, g_{k-1}$, and the local homology $C_{kn+1}^+(z_0^{\times kn})$ is non-trivial for infinitely many $n \in \mathbb{N}$.

Proposition 3.33 ([Maz13][Rue85]). *Let $z = z_0^{\times k}$ be a critical point of g such that $C_{kn+1}(z^{\times n})$ is non-trivial for infinitely $n \in \mathbb{N}$. Then 1 is the only eigenvalue of $DF([z_0])$, and the blow-up rotation number $\rho(I, [z_0])$ is equal to 0 for any identity isotopy of F fixing $[z_0]$.*

Remark 3.34. In particular, a symplectically degenerate maximum satisfies the condition of the proposition, and hence is a degenerate fixed point of F. Moreover, the previous proposition is still valid if we replace $C_{kn+1}(z^{\times n})$ with $C_{kn+1}^+(z^{\times n})$, and a symplectically degenerate minimum is also a degenerate fixed point of F.

3.2.3 The index at a symplectically degenerate extremum

As in the previous subsection, let $F : \mathbb{T}^2 \to \mathbb{T}^2$ be a Hamiltonian diffeomorphism, and $F = F_{k-1} \circ \cdots \circ F_0$ be a factorization by Hamiltonian diffeomorphisms F_i which are $\mathcal{C}^1$-close to the identity. Let f_j be the lift of F_j to $\mathbb{R}^2$ that is $\mathcal{C}^1$-close to the identity, and g_j be a generating function of f_j, for $j = 0, \cdots, k-1$. As was recalled in the previous subsection, if z_0 is a symplectically degenerate extremum, then the blow-up rotation number $\rho(I, [z_0])$ is equal to 0 for any identity isotopy of F fixing $[z_0]$. We will prove the following Proposition 3.35, and then can deduce Theorem 1.4 as an immediate corollary of Theorem 1.1.

Proposition 3.35. *If z_0 is a symplectically degenerate extremum, then $i(F, [z_0])$ is equal to 1.*

We will only deal with the case where 0 is a symplectically degenerated maximum, the other case can be treated similarly. Let us begin by some lemmas.

Lemma 3.36. *Suppose that g is a (local or global) generating function of a diffeomorphism f, and that 0 is a local maximum of g such that the Hessian matrix of g at 0 is degenerate. Let $I = (f_t)$ be the identity isotopy of f induced by g as in Section 3.2.1, and $\theta(t)$ be a continuous function such that*

$$\frac{J_{f_t}(0)\begin{pmatrix}\cos\theta(0)\\ \sin\theta(0)\end{pmatrix}}{\|J_{f_t}(0)\begin{pmatrix}\cos\theta(0)\\ \sin\theta(0)\end{pmatrix}\|} = \begin{pmatrix}\cos\theta(t)\\ \sin\theta(t)\end{pmatrix}.$$

Then, one can deduce that $\theta(1) \geq \theta(0)$.

Proof. As in Section 3.2.1, we denote the Hessian of g at 0 by

$$\mathrm{Hess}(g)(0) = \begin{pmatrix} \varrho & \sigma \\ \sigma & \tau \end{pmatrix}.$$

Since 0 is a local maximal point of g, Hess$g(0)$ is negative semi-definite. So, we knows that

$$\varrho \le 0, \quad \tau \le 0, \quad \text{and } \varrho\tau - \sigma^2 = 0.$$

As was proved in Section 3.2.1, if (a, b) is a unit eigenvector of $\mathrm{Hess}(g)(0)$ corresponding to the eigenvalue 0 , then it is a common eigenvector of $J_{f_t}(0)$ corresponding to the eigenvalue 1. Recall that

$$\varrho a + \sigma b = 0, \quad \text{and } \sigma a + \tau b = 0.$$

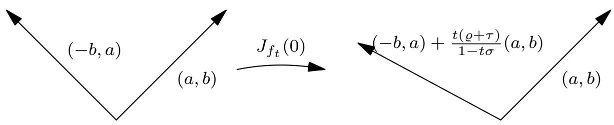
So,

$$J_{f_t}(0)\begin{pmatrix} -b \\ a \end{pmatrix} = \frac{1}{1-t\sigma}\begin{pmatrix} 1 & t\tau \\ -t\varrho & 1-2t\sigma \end{pmatrix}\begin{pmatrix} -b \\ a \end{pmatrix} = \begin{pmatrix} -b \\ a \end{pmatrix} + \frac{t(\varrho+\tau)}{1-t\sigma}\begin{pmatrix} a \\ b \end{pmatrix}.$$

Therefore

$$J_{f_t}(0)\Omega = \Omega\begin{pmatrix} 1 & \frac{t(\varrho+\tau)}{1-t\sigma} \\ 0 & 1 \end{pmatrix},$$

where $\Omega = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ is a normal matrix. Since $\frac{t(\varrho+\tau)}{1-t\sigma} \le 0$, one can deduce that $\theta(1) \ge \theta(0)$. □



Lemma 3.37. *If* 0 *is a symplectically degenerate maximum, then there exists a normal matrix* Ω *such that*

$$\Omega^{-1}J_{f_j}(0)\Omega = \begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix}$$

for $j = 0, \cdots, k-1$, *where* c_j *are non-positive real numbers.*

Proof. Let $I_j = (f_{j,t})_{t\in[0,1]}$ be the local isotopy of f_j induced by g_j as in section 3.2.1. Let $\mathcal{F}_j$ be the foliation whose leaves are the integral curves of the gradient vector field of g_j. As in Section 3.2.1, one can deduce that 0 is a sink of $\mathcal{F}_j$ and that $\mathcal{F}_j$ is locally transverse to I_j. Therefore, one knows that $\rho(I_j, 0) \ge 0$, and that $\rho(I_j, 0) = 0$ if and only if 0 is a degenerate fixed point of f_j.

Let $\theta : [0, k] \to \mathbb{R}$ be a continuous function such that

$$\frac{J_{f'_{jt}}(0)\begin{pmatrix} \cos\theta(j) \\ \sin\theta(j) \end{pmatrix}}{\|J_{f'_{jt}}(0)\begin{pmatrix} \cos\theta(j) \\ \sin\theta(j) \end{pmatrix}\|} = \begin{pmatrix} \cos\theta(j+t) \\ \sin\theta(j+t) \end{pmatrix}.$$

One knows that $\theta(j+1) > \theta(j)$ if $\rho(I_j, 0) > 0$, and $\theta(j+1) \ge \theta(j)$ if $\rho(I_j, 0) = 0$. But we know that $\rho(I_{k-1}\cdots I_0, z_0) = \rho(I, [z_0]) = 0$, so there exists $\theta(0) \in \mathbb{R}$ and a continuous function θ as above such that $\theta(k) = \theta(0)$. Therefore, $\rho(I_j, z_0) = 0$ for $j = 0, \cdots, k-1$ and $\begin{pmatrix}\cos\theta(0)\\ \sin\theta(0)\end{pmatrix}$ is a common eigenvector of $J_{f_j}(0)$ corresponding to the eigenvalue 1. As in the proof of the previous lemma, we can prove this lemma by choosing $\Omega = \begin{pmatrix}\cos\theta(0) & -\sin\theta(0)\\ \sin\theta(0) & \cos\theta(0)\end{pmatrix}$. $\square$

Lemma 3.38. *Suppose that g is a (local or global) generating function of a diffeomorphism f, that 0 is a local maximum of g, and that the Hessian matrix of g at 0 is degenerate. If Ω is a normal matrix, and if $f' = \Omega^{-1} f \Omega$ is generated by g' in a neighborhood of 0, then 0 is a local maximum of g' and $\mathrm{Hess}(g')(0)$ is degenerate.*

Proof. Since $\mathrm{Hess}(g)(0)$ is degenerate, 1 is an eigenvalue of $J_f(0)$ and hence an eigenvalue of $J_{f'}(0)$. So, $\mathrm{Hess}(g')(0)$ is degenerate.

Let $\mathcal{F}$ be the foliation whose leaves are integral curves of the gradient vector field of g, and $\mathcal{F}'$ be the foliation whose leaves are integral curves of the gradient vector field of g'. Let I_0 be a local isotopy of f satisfies $\rho(I_0, 0) = 0$, and I_0' be a local isotopy of f satisfies $\rho(I_0', 0) = 0$. As was proved in Section 3.2.1, $\mathcal{F}$ is locally transverse to I_0 and $\mathcal{F}'$ is locally transverse to I_0'. Therefore, $\Omega \circ \mathcal{F}'$ is locally transverse to I_0. Since 0 is a maximal point of g, it is a sink of $\mathcal{F}$. By the remark that follows Proposition 2.15, one deduces that 0 is a sink of $\Omega \circ \mathcal{F}'$, and hence a sink of $\mathcal{F}'$. Therefore, 0 is a local maximum of g'. $\square$

Lemma 3.39. *Let g_0 and g_1 be local generating functions of f_0 and f_1 respectively such that 0 is a local maximal point of both g_0 and g_1, and that the Hessian matrices satisfy*

$$Hess(g_i)(0) = \begin{pmatrix}0 & 0\\ 0 & c_i\end{pmatrix},$$

where $c_i \le 0$ for $i = 0, 1$. Then there exists a function g which is a generating function of $f = f_1 \circ f_0$ in a neighborhood of 0. Moreover, 0 is a local maximal point of g, and the Hessian matrix satisfies

$$Hess(g)(0) = \begin{pmatrix}0 & 0\\ 0 & c_0 + c_1\end{pmatrix}.$$

Proof. Suppose that $g_0(0) = g_1(0) = 0$. Since 0 is a local maximal point of both g_0 and g_1, it is a critical point of both g_0 and g_1. So,

$$\partial_1 g_0(0,0) = \partial_2 g_0(0,0) = \partial_1 g_1(0,0) = \partial_2 g_1(0,0) = 0.$$

Write $(x_1, y_1) = f_0(x_0, y_0)$ and $(x_2, y_2) = f_1(x_1, y_1)$. By definition of generating functions, one knows that

$$y_1 - y_0 + \partial_1 g_0(x_1, y_0) = 0, \quad \text{and } x_1 - x_2 + \partial_2 g_1(x_2, y_1) = 0. \tag{3.3}$$

Note that

$$\det\begin{pmatrix}\partial^2_{11} g_0(0,0) & 1\\ 1 & \partial^2_{22} g_1(0,0)\end{pmatrix} = -1.$$

So, by implicit function theorem, there exists a $\mathcal{C}^1$ diffeomorphism $\varphi: W \to W'$ such that $(x_1, y_1) = \varphi(x_2, y_0)$, where W and W' are sufficiently small neighborhoods of 0 in $\mathbb{R}^2$. Moreover,

$$J_\varphi(0,0) = -\begin{pmatrix}\partial^2_{11} g_0(0,0) & 1\\ 1 & \partial^2_{22} g_1(0,0)\end{pmatrix}^{-1} \begin{pmatrix}0 & \partial^2_{12} g_0(0,0) - 1\\ \partial^2_{12} g_1(0,0) - 1 & 0\end{pmatrix} = \begin{pmatrix}1 & -c_1\\ 0 & 1\end{pmatrix}.$$

Let

$$g(x_2, y_0) = g_0(x_1, y_0) + g_1(x_2, y_1) + (x_2 - x_1)(y_0 - y_1),$$

where $(x_1, y_1) = \varphi(x_2, y_0)$. We know that $g(0, 0) = 0$. In a neighborhood of 0, by a direct computation and equations (3.3), one knows that

$$\begin{aligned}\partial_1 g(x_2, y_0) =& \partial_1 g_0(x_1, y_0)\partial_1 x_1(x_2, y_0) + \partial_1 g_1(x_2, y_1) + \partial_2 g_1(x_2, y_1)\partial_1 y_1(x_2, y_0)\\ &+ (1 - \partial_1 x_1(x_2, y_0))(y_0 - y_1) - \partial_1 y_1(x_2, y_0)(x_2 - x_1)\\ =& \partial_1 g_0(x_1, y_0) + \partial_1 g_1(x_2, y_1).\end{aligned}$$

Similarly, one gets

$$\partial_2 g(x_2, y_0) = \partial_2 g_0(x_1, y_0) + \partial_2 g_1(x_2, y_1).$$

So, g is a $\mathcal{C}^2$ function near 0. Moreover,

$$\partial^2_{12} g(0, 0) = \partial^2_{11} g_0(0, 0)\partial_2 y_1(0, 0) + \partial^2_{12} g_0(0, 0) + \partial^2_{12} g_1(0, 0)\partial_2 y_1(0, 0) = 0.$$

Because g_0 and g_1 locally generate f_0 and f_1 respectively, one deduces

$$\partial_1 g(x_2, y_0) = -(y_2 - y_0) \quad \text{and} \quad \partial_2 g(x_2, y_0) = x_2 - x_0.$$

Therefore, g is a generating function of f in a neighborhood of 0.

By a direct computation, one gets

$$\partial^2_{11} g(0, 0) = \partial^2_{11} g_0(0, 0)\partial_1 x_1(0, 0) + \partial^2_{11} g_1(0, 0) + \partial^2_{12} g_1(0, 0)\partial_1 y_1(0, 0) = 0,$$

and

$$\partial^2_{22} g(0, 0) = \partial^2_{12} g_0(0, 0)\partial_2 x_1(0, 0) + \partial^2_{22} g_0(0, 0) + \partial^2_{22} g_1(0, 0) = c_0 + c_1.$$

So,

$$\mathrm{Hess}(g)(0) = \begin{pmatrix} 0 & 0 \\ 0 & c_0 + c_1 \end{pmatrix}.$$

We will conclude by proving that 0 is a locally maximum of g. Let $\varepsilon > 0$ be a small real number such that $|\varepsilon c_1| < 1$. We will prove that in a sufficiently small neighborhood of 0,

$$g(x_2, y_0) \leq g_0(x_1 + \frac{1}{\varepsilon}(y_0 - y_1), y_0) + g_1(x_2, y_1 + \varepsilon(x_2 - x_1)) \leq 0,$$

and hence 0 is a locally maximum of g because the second inequality is strict for $(x_2, y_0) \neq 0$. Indeed, by Taylor's theorem and equations (3.3), one knows that in a sufficiently small neighborhood of 0,

$$\begin{aligned}g_0(x_1 + \frac{1}{\varepsilon}(y_0 - y_1), y_0) &= g_0(x_1, y_0) + \frac{1}{\varepsilon}\partial_1 g_0(x_1, y_0)(y_0 - y_1) + \frac{1}{2\varepsilon^2}\partial^2_{11} g_0(\xi, y_0)(y_0 - y_1)^2\\ &= g_0(x_1, y_0) + \frac{1}{\varepsilon}(y_0 - y_1)^2 + \frac{1}{2\varepsilon^2}\partial^2_{11} g_0(\xi, y_0)(y_0 - y_1)^2,\end{aligned}$$

where ξ is a real number between x_1 and $x_1 + \frac{1}{\varepsilon}(y_0 - y_1)$. Similarly, one deduces that in sufficiently small neighborhood of 0,

$$g_1(x_2, y_1 + \varepsilon(x_2 - x_1)) = g_1(x_2, y_1) + \varepsilon(x_2 - x_1)^2 + \frac{\varepsilon^2}{2}\partial^2_{22} g_1(x_2, \eta)(x_2 - x_1)^2,$$

where η is a real number between y_1 and $y_1 + \varepsilon(x_2 - x_1)$. So,

$$\begin{aligned} g(x_2, y_0) =& g_0(x_1 + \frac{1}{\varepsilon}(y_0 - y_1), y_0) + g_1(x_2, y_1 + \varepsilon(x_2 - x_1)) \\ & - \frac{1}{2\varepsilon}(y_0 - y_1)^2 - \frac{\varepsilon}{2}(x_2 - x_1)^2 + (x_2 - x_1)(y_0 - y_1) \\ & - \frac{1}{2\varepsilon}(1 + \frac{1}{\varepsilon}\partial_{11} g_0(\xi, y_0))(y_0 - y_1)^2 - \frac{\varepsilon}{2}(1 + \varepsilon \partial_{22} g_1(x_2, \eta))(x_2 - x_1)^2. \end{aligned}$$

For $(x_2, y_0) \neq 0$ that is in a sufficiently small neighborhood of 0, one can suppose that

$$|\frac{1}{\varepsilon^2}\partial_{11} g_0(\xi, y_0)| < 1, \quad \text{and } |\varepsilon \partial_{22} g_1(x_2, \eta)| < 1.$$

So,

$$g(x_2, y_0) \leq g_0(x_1 + \frac{1}{\varepsilon}(y_0 - y_1), y_0) + g_1(x_2, y_1 + \varepsilon(x_2 - x_1)).$$

□

Now, we begin the proof of Proposition 3.35.

Proof of Propostion 3.35. Suppose that z_0 is a symplectically degenerated maximum. By Lemma 3.37, there exists a coordinate transformation such that in the new coordinate system the Jacobian of each f_j at z_0 has the form

$$\begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix}$$

where c_j is a non-positive real number. We consider everything in the new coordinate system. Each f_j can be locally generated by a generating function g'_j, and the Hessian of g'_j at z_0 has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & c_j \end{pmatrix}.$$

By Lemma 3.38, z_0 is a local maximum of each g'_j. So, by Lemma 3.39, we can construct a generating function g' such that

- z_0 is a local maximum of g',
- $\text{Hess}(g')(z_0)$ is degenerate,
- g' generates $f = f_{k-1} \cdots f_0$ in a neighborhood of z_0.

So, by Proposition 3.26, we know $i(f, z_0)$ is equal to 1, and hence $i(F, [z_0])$ is equal to 1. □

Chapter 4

Torsion-low isotopies

4.1 The rotation type at an isolated fixed point of an orientation preserving local homeomorphism

Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at the isolated fixed point $0 \in \mathbb{R}$. The main aim of this section is to detect the local rotation type of the local isotopies of f and prove Theorem 1.5.

Before proving the theorem, we will first prove the following lemma:

Lemma 4.1. *If f satisfies the local intersection condition, then a local isotopy $I = (f_t)_{t\in[0,1]}$ of f can not have both a positive and a negative rotation type.*

Proof. We will give a proof by contradiction. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are two locally transverse foliations of I such that 0 is a sink of $\mathcal{F}_1$ and a source of $\mathcal{F}_2$. Then, there exist two orientation preserving local homeomorphisms $h_1 : (V_1, 0) \to (\mathbb{D}, 0)$ and $h_2 : (V_2, 0) \to (\mathbb{D}, 0)$ such that h_1 (resp. h_2) sends the restricted foliation $\mathcal{F}_1|_{V_1}$ (resp. $\mathcal{F}_2|_{V_2}$) to the radial foliation on $\mathbb{D}$ with the orientation toward (resp. backward) 0, where $\mathbb{D}$ is the unit disk centered at 0, and $V_i \subset W$ is a small neighborhood of 0 such that f does not have any fixed point in V_i except 0, and $f(\gamma)\cap\gamma \neq \emptyset$ for all essential closed curve γ in $V_i\setminus\{0\}$, for $i = 1, 2$. We denote by $\mathbb{D}_r$ the disk centered at 0 with radius r, and S_r the boundary of $\mathbb{D}_r$. Choose $0 < r_2 < 1$ such that for all $z \in h_2^{-1}(S_{r_2})$, there exists an arc in $V_2 \setminus \{0\}$ that is homotopic to $t \mapsto f_t(z)$ in $V_2 \setminus \{0\}$ and is positively transverse to $\mathcal{F}_2$; choose $0 < r'_2 < r_2$ such that $h_2 \circ f \circ h_2^{-1}(S_{r_2}) \subset \mathbb{D}\setminus\mathbb{D}_{r'_2}$; choose $0 < r'_1 < r'_2$ such that $h_1^{-1}(\overline{\mathbb{D}_{r'_1}}) \subset h_2^{-1}(\mathbb{D}_{r'_2})$; and choose $0 < r_1 < r'_1$ such that $h_1 \circ f \circ h_1^{-1}(\mathbb{D}_{r_1}) \subset \mathbb{D}_{r'_1}$, and for all $z \in h_1^{-1}(S_{r_1})$, there exists an arc in $V_1 \setminus \{0\}$ that is homotopic to $t \mapsto f_t(z)$ in $V_1 \setminus \{0\}$ and is positively transverse to $\mathcal{F}_1$. We consider a homeomorphism $h : (V_2, 0) \to (\mathbb{D}, 0)$ such that $h|_{h_1^{-1}(\mathbb{D}_{r'_1})} = h_1$ and $h|_{h_2^{-1}(\mathbb{D}\setminus\mathbb{D}_{r'_2})} = h_2$. Then, $h \circ f \circ h^{-1}$ does not have any fixed point except 0. Let

$$\begin{array}{rcl} \pi : \mathbb{R} \times (-\infty, 0) & \to & \mathbb{R}^2 \setminus \{0\} \simeq \mathbb{C} \setminus \{0\} \\ (\theta, y) & \mapsto & -ye^{i2\pi\theta} \end{array}$$

be the universal covering projection, and $\widetilde{f}'$ be the lift of $h \circ f \circ h^{-1}$ associated to $I' = (h \circ f_t \circ h^{-1})_{t\in[0,1]}$. Then, $p_1(\widetilde{f}'(\theta, -r_1)) - \theta > 0$ and $p_1(\widetilde{f}'(\theta, -r_2)) - \theta < 0$ for all $\theta \in \mathbb{R}$, where p_1 is the projection onto the first factor. Then, $h \circ f \circ h^{-1}$ is a map satisfying the conditions of Proposition 2.27. But we know that $h \circ f \circ h^{-1}(\gamma) \cap \gamma \neq \emptyset$ for all essential simple closed curve γ in $\mathbb{D} \setminus \{0\}$, which is a contradiction. □

Remark 4.2. In particular, a local homeomorphism satisfying the assumption of Theorem 1.5 also satisfies the condition of the previous lemma. But not all local isotopies can not have both a positive and a negative rotation type. As we can see in Section 4.3, there exist local isotopies that have both positive and negative rotation types.

Now, we begin the proof of Theorem 1.5.

Proof of Theorem 1.5 . To simplify the notations, we suppose that the local homeomorphism is at $0 \in \mathbb{R}^2$. One has to consider two cases: $i(f, 0)$ is equal to 1 or not.

a) Suppose that $i(f, 0) \neq 1$. By Proposition 2.1, there exists a unique homotopy class of local isotopies at 0 such that $i(I_0, 0) = i(f, 0) - 1 \neq 0$ for every local isotopy I_0 in this class. Let $\mathcal{F}$ be a locally transverse foliation of I_0. Then $i(\mathcal{F}, 0) = i(I_0, 0) + 1 \neq 1$ by Proposition 2.3, and therefore 0 is neither a sink nor a source of $\mathcal{F}$. This implies that I_0 has neither a positive nor a negative rotation type. So, I_0 has a zero rotation type at z_0. For a local isotopy I at 0 that is not in the homotopy class of I_0, by Proposition 2.14, it has only a positive rotation type if $I > I_0$, and has only a negative rotation type if $I < I_0$. Then, both statements of Theorem 1.5 are proved.

b) Suppose that $i(f, 0) = 1$. Let I be a local isotopy of f, and $\mathcal{F}$ be an oriented foliation that is locally transverse to I. Since there exists a neighborhood $U \subset W$ of 0 that contains neither the positive nor the negative orbit of any wandering open set, one knows (see the remark following Proposition 2.12) that 0 is either a sink, a source or a saddle of $\mathcal{F}$. As recalled in Proposition 2.12, in the first two cases $i(\mathcal{F}, 0)$ is equal to 1, and in the last case $i(\mathcal{F}, 0)$ is not positive. By Proposition 2.3 one deduces that $i(\mathcal{F}, 0) = 1$ because $i(f, 0) = 1$. So, 0 is a sink or a source. Therefore, I has exactly one of the three rotation types by Lemma 4.1.

Since there exists a neighborhood $U \subset W$ of 0 that contains neither the positive nor the negative orbit of any wandering open set, one deduces by Proposition 2.18 that $\rho_s(I, 0)$ is not empty, and knows that f satisfies the local intersection condition. Moreover, 0 is an isolated fixed point, so one can deduce by the first three assertions of Proposition 2.20 that there exists $k \in \mathbb{Z}$ such that $\rho_s(I, 0)$ is a subset of $[k, k + 1]$. By the assertion i) of Proposition 2.20, there exists a local isotopy I_0 of f such that $\rho_s(I_0, 0)$ is a nonempty subset of $[0, 1]$ and is not reduced to 1. Then, as a corollary of the assertions iv)-vi) of Proposition 2.20,

- I has a positive rotation type if $I > I_0$,
- I has a negative rotation type if $I < I_0$.

□

Remark 4.3. It is easy to see that the condition that there exists a neighborhood $U \subset W$ of 0 that contains neither the positive nor the negative orbit of any wandering open set is necessary for the first assertion of the theorem. Indeed, if we do not require this condition, even if f satisfies the local intersection condition, there still exists local isotopies that have both positive (resp. negative) and zero rotation types. We will give one such example in Section 4.3.

Remark 4.4. Matsumoto [Mat01] defined a notion of positive and negative type for an orientation and area preserving local homeomorphism at an isolated fixed point with Lefschetz index 1. In this case, our definitions of "positive rotation type" (resp. "negative rotation type") is equivalent to his definition of "positive type" (resp. negative type").

Now, let us prove Proposition 1.6.

Proof of Proposition 1.6. The first statement is just a corollary of the definition of the torsion-low property and the assertions i), iv) of Proposition 2.20. Suppose now that f can be blown-up at z_0. If f satisfies the hypothesis, $\rho_s(I, z_0)$ is not empty by Proposition 2.18. So, using the assertion vii) of Proposition 2.20, one deduces that $\rho_s(I, z_0)$ is reduced to a single point in $[-1, 1]$. Suppose now that f is a diffeomorphism in a neighborhood of z_0. The first part of the third statement is just a special case of the second statement.

To conclude, let us prove the last part of the third statement. To simplify the notations, we suppose that $z_0 = 0 \in \mathbb{R}^2$. Since there exists a neighborhood of 0 that contains neither the positive nor the negative orbit of any wandering open set, $Df(0)$ can not have two real eigenvalues such that the absolute values of both eigenvalues are strictly smaller (resp. bigger) than 1. Since 1 is not an eigenvalue of $Df(0)$, one has to consider the following three cases:

- Suppose that $Df(0)$ do not have any real eigenvalue. In this case, $\rho(I, 0)$ is not an integer.
- Suppose that $Df(0)$ has two real eigenvalues λ_1 and λ_2 such that $\lambda_1 < -1 < \lambda_2 < 0$. In this case, $\rho(I, 0)$ is equal to $\frac{1}{2}$ or $-\frac{1}{2}$, and is not an integer.
- Suppose that $Df(0)$ has two real eigenvalues λ_1 and λ_2 such that $0 < \lambda_1 < 1 < \lambda_2$. In this case, $i(f, 0) = -1$, and I has a zero rotation type at 0. So, $\rho(I, 0)$ is equal to 0.

Anyway, we know that $\rho(I, 0)$ belongs to $(-1, 1)$. □

4.2 The existence of a global torsion-low isotopy

Let f be an orientation and area preserving homeomorphism of a connected oriented surface M that is isotopic to the identity. The main aim of this section is to prove the existence of a torsion-low maximal isotopy of f, i.e. Theorem 1.7.

When $\mathrm{Fix}(f) = \emptyset$, the theorem is trivial, and so we suppose that $\mathrm{Fix}(f) \neq \emptyset$ in the following part of this section. Recall that $\mathcal{I}$ is the set of couples (X, I_X) that consists of a closed subset $X \subset \mathrm{Fix}(f)$ and an identity isotopy I_X of f on M that fixes all the points in X. We denote by $\mathcal{I}_0$ be the set of $(X, I_X) \in \mathcal{I}$ such that I_X is torsion-low at every $z \in X$. Recall that $\precsim$ is Jaulent's preorder defined in Section 2.6. Then, Theorem 1.7 is just an immediate corollary of the following theorem. Moreover, the proof do not need any other assumptions when $\mathrm{Fix}(f)$ is totally disconnected, while we should admit the yet unpublished results of Béguin, Le Roux and Crovisier stated in Section 2.6 when $\mathrm{Fix}(f)$ is not totally disconnected.

Theorem 4.5. *Given $(X, I_X) \in \mathcal{I}_0$, there exists a maximal extension $(X', I_{X'})$ of (X, I_X) that belongs to $\mathcal{I}_0$.*

Remark 4.6. We will see that, except in the case where M is a sphere and X is reduced to a point, $I_{X'}$ and I_X are equivalent as local isotopies at z, for every $z \in X$. In the case where M is a sphere and X is reduced to one point, this is not necessary the case. We will give an example in Section 4.3.

Remark 4.7. One may fail to find a torsion-low maximal identity isotopy I such that $0 \in \rho_s(I, z)$ for every $z \in \mathrm{Fix}(I)$ that is not isolated in $\mathrm{Fix}(f)$. We will give an example in Section 4.3. In particular, in this example, for every torsion-low maximal identity isotopy, there is a point that is isolated in $\mathrm{Fix}(I)$ but is not isolated in $\mathrm{Fix}(f)$.

Before proving this theorem, we will first state some properties of a torsion-low maximal isotopy.

Proposition (Proposition 1.10)**.** *Let f be an area preserving homeomorphism of M that is isotopic to the identity, I be a maximal identity isotopy that is torsion-low at $z \in \mathrm{Fix}(I)$, and $\mathcal{F}$ be a transverse foliation of I. If z is an isolated singularity of $\mathcal{F}$, then*

- *z is a saddle of $\mathcal{F}$ and $i(\mathcal{F}, z) = i(f, z)$, if z is an isolated fixed point of f such that $i(f, z) \neq 1$;*
- *z is a sink or a source of $\mathcal{F}$ if z is an isolated fixed point such that $i(f, z) = 1$ or if z is not isolated in* $\mathrm{Fix}(f)$*.*

Proof. One has to consider two cases: z is isolated in $\mathrm{Fix}(f)$ or not.

i) Suppose that z is isolated in $\mathrm{Fix}(f)$, then as a corollary of Theorem 1.5,
  - z is neither a sink nor a source of $\mathcal{F}$ if $i(f, z) \neq 1$;
  - z is a sink or a source of $\mathcal{F}$ if $i(f, z) = 1$.

  Moreover, in the first case, z is a saddle of $\mathcal{F}$ and $i(\mathcal{F}, z) = i(f, z)$ by Proposition 2.3 and the remark that follows Proposition 2.12.

ii) Suppose that z is not isolated in $\mathrm{Fix}(f)$. Let D be a small closed disk containing z as an interior point such that D does not contain any other fixed point of I, and $V \subset D$ be a neighborhood of z such that for every $z' \in V$, the trajectory of z' along I is contained in D. We define the rotation number of a fixed point $z' \in V \setminus \{z\}$ to be the integer k such that its trajectory along I is homotopic to $k\partial D$ in $D \setminus \{z\}$. Then, by the maximality of I, the rotation number of a fixed point $z' \in V \setminus \{z\}$ is nonzero, and 0 is not an interior point of the convex hull of $\rho_s(I, z)$, as tells us the assertion iii) of Proposition 2.20. Since z is accumulated by fixed points of f, there exist $k_0 \subset \mathbb{Z} \setminus \{0\} \cup \{\pm\infty\}$ and a sequence of fixed points $\{z_n\}_{n\in\mathbb{N}}$ converging to z, such that their rotation numbers converge to k_0. Then, k_0 belongs to $\rho_s(I, z)$.

  When $k_0 > 0$, $\rho_s(I, z)$ is included in $[0, +\infty]$ and not reduced to 0. By the assertion v) of Proposition 2.20, one deduces that z is a sink of $\mathcal{F}$. For the same reason, when $k_0 < 0$, we deduce that z is a source of $\mathcal{F}$.

□

The following result is an immediate corollary of Theorem 1.7 and Proposition 1.6.

Corollary 4.8 (Proposition 1.11)**.** *Let f be an area preserving diffeomorphism of M that is isotopic to the identity. Then, there exists a maximal isotopy I, such that for all $z \in \mathrm{Fix}(I)$, the rotation number satisfies*

$$-1 \leq \rho(I, z) \leq 1.$$

Moreover, the inequalities are both strict if z is not degenerate.

Remark 4.9. One may fail to get the strict inequalities without the assumption of non-degenerality. We will give an example in Section 4.3.

Now, we begin the proof of Theorem 4.5. We first note the following fact which results immediately from the definition:

If $(Y, I_Y) \in \mathcal{I}$ and $z \in Y$ is a point such that I_Y is not torsion-low at z, then z is isolated in Y.

Then, given such a couple $(Y, I_Y) \in \mathcal{I}$, we will try to find an extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and $z' \in Y' \setminus (Y \setminus \{z\})$ such that $I_{Y'}$ is torsion-low at z'.

We will divide the proof into two cases. Unlike the second case, the first case does not use the result of Béguin, Le Roux and Crovisier stated in Section 2.6, but only use Jaulent's results.

4.2.1 Proof of Theorem 4.5 when $\mathrm{Fix}(f)$ is totally disconnected

We suppose that $\mathrm{Fix}(f)$ is totally disconnected in this subsection. In this case, Theorem 4.5 is a corollary of Zorn's lemma and the following Propositions 4.10-4.13. We will explain first why the propositions imply the theorem, then we will prove the four propositions one by one. We will also give a proof of Proposition 1.9 at the end of this subsection.

Proposition 4.10. *If $\{(X_\alpha, I_{X_\alpha})\}_{\alpha\in J}$ is a totally ordered chain in $\mathcal{I}_0$, then there exists an upper bound $(X_\infty, I_{X_\infty}) \in \mathcal{I}_0$ of this chain, where $X_\infty = \overline{\cup_{\alpha\in J} X_\alpha}$*

Proposition 4.11. *For every maximal $(Y, I_Y) \in \mathcal{I}$ and $z \in Y$ such that I_Y is not torsion-low at z and $M\setminus(Y\setminus\{z\})$ is neither a sphere nor a plane, there exist a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of $(Y\setminus\{z\}, I_Y)$ and $z' \in Y'\setminus(Y\setminus\{z\})$ such that $I_{Y'}$ is torsion-low at z'.*

Proposition 4.12. *When M is a plane and f admits a fixed point, $(X, I_X) \in \mathcal{I}_0$ is not maximal in $(\mathcal{I}_0, \precsim)$ if $X = \emptyset$.*

Proposition 4.13. *When M is a sphere, $(X, I_X) \in \mathcal{I}_0$ is not maximal in $(\mathcal{I}_0, \precsim)$ if $\#X \leq 1$.*

Remark 4.14. Proposition 4.12 and 4.13 deal with two special cases. The first is easy, while the second is more difficult. Indeed, to find an identity isotopy on a plane that is torsion-low at one point, we do not need to know the dynamics at infinity; but to find an identity isotopy on a sphere that is torsion-low at two points, we need check the properties of the isotopy near both points.

Proof of Theorem 4.5 when $\mathrm{Fix}(f)$ *is totally disconnected.* Fix $(X, I_X) \in \mathcal{I}_0$. Let $\mathcal{I}_*$ be the set of equivalent classes of the extensions $(X', I_{X'}) \in \mathcal{I}_0$ of (X, I_X). Then, the pre-order $\precsim$ induces a partial order over $\mathcal{I}_*$. To simplify the notations, we still denote by $\precsim$ this partial order. By Proposition 4.10, $(\mathcal{I}_*, \precsim)$ is a partial ordered set satisfying the condition of Zorn's lemma, so $(\mathcal{I}_*, \precsim)$ contains at least one maximal element. Choose one representative $(X', I_{X'})$ of a maximal element of $(\mathcal{I}_*, \precsim)$. It is an extension of (X, I_X) and is maximal in $(\mathcal{I}_0, \precsim)$.

Using Proposition 4.11-4.13, we will prove by contradiction that a maximal couple $(X, I_X) \in (\mathcal{I}_0, \precsim)$ is also maximal in $(\mathcal{I}, \precsim)$. Suppose that there exists a couple $(X, I_X) \in \mathcal{I}_0$ that is maximal in $(\mathcal{I}_0, \precsim)$ but is not maximal in $(\mathcal{I}, \precsim)$. Fix a maximal extension (Y, I_Y) of (X, I_X) in $(\mathcal{I}, \precsim)$, and $z \in Y\setminus X$. Then, I_Y is not torsion-low at z, and so z is isolated in Y. Write $Y_0 = Y\setminus\{z\}$. By Proposition 4.12 and 4.13, $M\setminus Y_0$ is neither a sphere nor a plane. By Proposition 4.11, there exist a maximal extension $(Y', I_{Y'})$ of (Y_0, I_Y) and $z' \in Y'$, such that $I_{Y'}$ is torsion-low at z'. Then $(X\cup\{z'\}, I_{Y'}) \in \mathcal{I}_0$ is an extension of (X, I_X), which contradicts the maximality of (X, I_X) in $(\mathcal{I}_0, \precsim)$. □

Proof of Proposition 4.10. By Proposition 2.7, we know that there exists an upper bound $(X_\infty, I_{X_\infty}) \in \mathcal{I}$ of the chain, where $X_\infty = \overline{\cup_{\alpha\in J} X_\alpha}$. We only need to prove that $(X_\infty, I_{X_\infty}) \in \mathcal{I}_0$.

When J is finite, the result is obvious. We suppose that J is infinite. Fix $z \in X_\infty$. Either it is a limit point of X_∞, or there exists $\alpha_0 \in J$ such that z is an isolated point of X_α for all $\alpha \in J$ satisfying $(X_{\alpha_0}, I_{X_{\alpha_0}}) \precsim (X_\alpha, I_{X\alpha})$. In the first case, $0 \in \rho_s(I_{X_\infty}, z)$; in the second case, I_{X_∞} is locally homotopic to $I_{X_{\alpha_0}}$ at z. In both case, I_{X_∞} is torsion-low at z. □

Before proving Proposition 4.11, we will first prove the following two lemmas (Lemma 4.15 and 4.16). We will use Lemma 4.15 when proving Lemma 4.16, and we will use Lemma 4.16 when proving Proposition 4.11.

Lemma 4.15. *Let us suppose that (Y, I_Y) is maximal in $(\mathcal{I}, \precsim)$, that I_Y is not torsion-low at $z \in Y$, and that $M \setminus (Y \setminus \{z\})$ is neither a sphere nor a plane. If for every maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and every point $z' \in Y' \setminus (Y \setminus \{z\})$, $I_{Y'}$ is not torsion-low at z', then there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of $(Y \setminus \{z\}, I_Y)$ such that $\#(Y' \setminus (Y \setminus \{z\})) > 1$.*

Proof. Fix a couple (Y, I_Y) maximal in $(\mathcal{I}, \precsim)$ and $z_0 \in Y$ satisfying the assumptions of this lemma. Then, z_0 is an isolated point of Y. Write $Y_0 = Y \setminus \{z_0\}$. Then Y_0 is a closed subset and $M \setminus Y_0$ is neither a sphere nor a plane. Due to Remark 2.21, one has to consider the following four cases:

i) z_0 is an isolated fixed point of f and there exists a local isotopy $I'_{z_0} > I_Y$ at z_0 which does not have a positive rotation type;
ii) z_0 is not an isolated fixed point of f and $\rho_s(I_Y, z_0) \subset [-\infty, -1)$;
iii) z_0 is an isolated fixed point of f and there exists a local isotopy $I'_{z_0} < I_Y$ at z_0 which does not have a negative rotation type;
iv) z_0 is not an isolated fixed point of f and $\rho_s(I_Y, z_0) \subset (1, +\infty]$.

We will study the first two cases, the other ones can be treated in a similar way.

Let $\mathcal{F}_Y$ be a transverse foliation of I_Y. In case i), by Theorem 1.5, there exists a local isotopy I_0 at z_0 that is torsion-low at z_0, and we know that $I_Y < I'_{z_0} \precsim I_0$, so I_Y has a negative rotation type at z_0; in case ii), we know that I_Y has a negative rotation type at z_0 by the assertion v) of Proposition 2.20 and the fact that $\rho_s(I_Y, z_0) \subset [-\infty, -1)$. Anyway, z_0 is a source of $\mathcal{F}_Y$. We denote by W the repelling basin of z_0 for $\mathcal{F}_Y$.

Let $\pi_{Y_0} : \widetilde{M}_{Y_0} \to M \setminus Y_0$ be the universal cover, $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ be the identity isotopy that lifts $I_Y|_{M\setminus Y_0}$, $\widetilde{f} = \widetilde{f}_1$ be the induced lift of $f|_{M\setminus Y_0}$, and $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}_Y$. Then, $\widetilde{I}$ fixes every point in $\pi_{Y_0}^{-1}\{z_0\}$, and every point in $\pi_{Y_0}^{-1}\{z_0\}$ is a source of $\widetilde{\mathcal{F}}$. We fix one element $\widetilde{z}_0$ in $\pi_{Y_0}^{-1}\{z_0\}$, and denote by $\widetilde{W}$ the repelling basin of $\widetilde{z}_0$ for $\widetilde{\mathcal{F}}$. Let $J_{\widetilde{z}_0}$ be an identity isotopy of the identity map of $\widetilde{M}_{Y_0}$ that fixes $\widetilde{z}_0$ and satisfies $\rho_s(J_{\widetilde{z}_0}, \widetilde{z}_0) = \{1\}$. Let $\widetilde{I}^*$ be a maximal extension of $(\{\widetilde{z}_0\}, J_{\widetilde{z}_0}\widetilde{I})$, and $\widetilde{\mathcal{F}}^*$ be a transverse foliation of $\widetilde{I}^*$.

Because $M \setminus Y_0$ is neither a sphere not a plane, $\pi_{Y_0}^{-1}\{z_0\}$ is not reduced to one point, and $\widetilde{W}$ is a proper subset of $\widetilde{M}_{Y_0}$. Moreover, if we consider the end ∞ as a singularity, the disk bounded by the union of $\{\infty\}$ and a leaf of $\widetilde{\mathcal{F}}$ in the boundary of $\widetilde{W}$ is a petal. Consequently, $\widetilde{f}$ can be blown-up at ∞ by the criteria in Section 2.10. On the other hand, ∞ is accumulated by the points of $\pi_{Y_0}^{-1}\{z_0\}$, so 0 belongs to $\rho_s(\widetilde{I}, \infty)$. Therefore, $\rho_s(\widetilde{I}, \infty)$ is reduced to 0 by the assertion vii) of Proposition 2.20, and $\rho_s(\widetilde{I}^*, \infty)$ is reduced to -1 by the first assertion of Proposition 2.20.

We can assert that $\widetilde{I}^*$ has finitely many fixed points. We will prove it by contradiction. Suppose that $\widetilde{I}^*$ fixes infinitely many points. Because $\rho_s(\widetilde{I}^*, \infty)$ is reduced to -1, ∞ is not accumulated by fixed points of $\widetilde{I}^*$. Since $\widetilde{I}$ fixes each point in $\pi_{Y_0}^{-1}\{z_0\}$, $\widetilde{I}^*$ does not fix any point in $\pi_{Y_0}^{-1}\{z_0\} \setminus \{\widetilde{z}_0\}$. Since I_Y is not torsion-low at z_0, $\widetilde{z}_0$ is isolated in $\mathrm{Fix}(\widetilde{I}^*)$ (otherwise, z_0 is accumulated by fixed points of f and $-1 \in \rho_s(I_Y, z_0)$). Therefore, there exists a non-isolated point $\widetilde{z}'$ in $\mathrm{Fix}(\widetilde{I}^*)$ such that $z' = \pi_{Y_0}(\widetilde{z}') \neq z_0$, and one knows that 0 belongs to $\rho_s(\widetilde{I}^*, \widetilde{z}')$. Moreover, z' is a non-isolated fixed point of f. By Proposition 2.6, there exists an extension $(Y', I_{Y'})$ of (Y_0, I_Y) that fixes z'. Let $\widetilde{I}'$ be the identity isotopy that lifts $I_{Y'}|_{M\setminus Y_0}$. Since $\pi_{Y_0}^{-1}(z')$ is included in $\mathrm{Fix}(\widetilde{I}')$, we have $\rho_s(\widetilde{I}', \infty) = 0$. Therefore, $\widetilde{I}'$ and $J_{\widetilde{z}'}^{-1}\widetilde{I}^*$ are equivalent as local isotopies at ∞, where $J_{\widetilde{z}'}$ is an identity isotopy of the identity map of $\widetilde{M}_{Y_0}$ that fixes $\widetilde{z}'$ and satisfies $\rho_s(J_{\widetilde{z}'}, \widetilde{z}') = \{1\}$. Recall that $\pi_1(\mathrm{homeo}_0(\mathbb{R}^2, 0)) \cong \mathbb{Z}$, so $\widetilde{I}'$ and $J_{\widetilde{z}'}^{-1}\widetilde{I}^*$ are also equivalent as local isotopies at $\widetilde{z}'$ (see Section 2.3), which means that -1 belongs to $\rho_s(I_{Y'}, z')$. So, $I_{Y'}$ is torsion-low at z', which contradicts the assumption of this lemma.

Since $\rho_s(\widetilde{I}^*,\infty)$ is reduced to -1, the assertion v) of Proposition 2.20 tells us that ∞ is a source of $\widetilde{\mathcal{F}}^*$. We can assert that $\widetilde{z}_0$ is not a sink of $\widetilde{\mathcal{F}}^*$. Indeed, in case i), one knows that $\widetilde{I}^* \precsim I'_{z_0}$ as a local isotopy at z_0, and that I'_{z_0} does not have a positive rotation type, so $\widetilde{I}^*$ does not have a positive rotation type; in case ii), one knows that $\rho_s(\widetilde{I}^*,\widetilde{z}_0) = \rho_s(J_{\widetilde{z}_0}\widetilde{I},\widetilde{z}_0) \subset [-\infty,0)$, and the result is a corollary of the assertion v) of Proposition 2.20.

In $\widetilde{M}_{Y_0} \sqcup \{\infty\}$, there does not exist any closed leaf or oriented simple closed curve that consists of leaves and singularities of $\widetilde{\mathcal{F}}^*$ with the orientation inherited from the orientation of leaves. We can prove this assertion by contradiction. Let Γ be such a curve. Since ∞ is a source of $\widetilde{\mathcal{F}}^*$, it does not belong to Γ. Let U be the bounded component of $\widetilde{M}_{Y_0} \setminus \Gamma$, then U contains the positive or the negative orbit of a wandering open set in $U \setminus \widetilde{f}(U)$ or $U \setminus \widetilde{f}^{-1}(U)$ respectively. This contradicts the area preserving property of $\widetilde{f}$.

Then, we can give a partial order $<$ over the set of singularities of $\widetilde{\mathcal{F}}^*$ such that $\widetilde{z} < \widetilde{z}'$ if there exists a leaf or a connection of leaves and singularities with the orientation inherited from the orientation of leaves from $\widetilde{z}'$ to $\widetilde{z}$. Since $\widetilde{\mathcal{F}}^*$ has only finitely many singularities, there exists a minimal singularity $\widetilde{z}_1$. Moreover, $\widetilde{F}^*$ does not have any closed leaf or a leaf from $\widetilde{z}_1$, and hence $\widetilde{z}_1$ is a sink of $\widetilde{\mathcal{F}}^*$. Therefore, $\widetilde{f}$ fixes $\widetilde{z}_1$ and hence there exists a maximal extension (Y_1, I_{Y_1}) of (Y_0, I_Y) such that $Y_0 \cup \{z_1\} \subset Y_1$, where $z_1 = \pi_{Y_0}(\widetilde{z}_1)$.

Now, we will prove by contradiction that $Y_1 \setminus Y_0$ contains at least two points. Suppose that $Y_1 = Y_0 \sqcup \{z_1\}$. Let $\mathcal{F}_{Y_1}$ be a transverse foliation of I_{Y_1}, $\widetilde{I}_1$ be the identity isotopy that lifts $I_{Y_1}|_{M\setminus Y_0}$, and $\widetilde{\mathcal{F}}_1$ be the lift of $\mathcal{F}_{Y_1}$ to $\widetilde{M}_{Y_0}$. We know that $(Y_0, I_Y) \sim (Y_0, I_{Y_1})$, so the lift of $f|_{M\setminus Y_0}$ to $\widetilde{M}_{Y_0}$ associated to I_{Y_1} is also $\widetilde{f}$. The set of singularities of $\widetilde{\mathcal{F}}_1$ is $\pi_{Y_0}^{-1}\{z_1\}$, and $\widetilde{z}_1$ is an isolated singularity of $\widetilde{\mathcal{F}}_1$, so it is a sink, or a source, or a saddle of $\widetilde{\mathcal{F}}_1$ by Remark 2.13. We know that $\rho_s(\widetilde{I}^*,\infty)$ is reduced to -1 and that $\rho_s(\widetilde{I}_1,\infty)$ is reduced to 0, so $\widetilde{I}^*$ and $J_{\widetilde{z}_1}\widetilde{I}_1$ are equivalent as local isotopies at $\widetilde{z}_1$. By the assumption, I_{Y_1} is not torsion-low at z_1, so $\widetilde{z}_1$ is a sink of $\widetilde{\mathcal{F}}_1$, and z_1 is a sink of $\mathcal{F}_{Y_1}$. Let $\widetilde{W}_1$ be the attracting basin of $\widetilde{z}_1$ for $\widetilde{\mathcal{F}}_1$. A leaf in $\partial\widetilde{W}_1$ is a proper leaf. For every fixed point $\widetilde{z}$ of $\widetilde{f}$, there exists a loop δ that is homotopic to its trajectory along $\widetilde{I}_1$ in $\widetilde{M}_{Y_0} \setminus \pi_{Y_0}^{-1}\{z_1\}$ (so in $\widetilde{M}_{Y_0} \setminus \{z_1\}$) and is transverse to $\widetilde{\mathcal{F}}_1$. The linking number $L(\widetilde{I}_1,\widetilde{z},\widetilde{z}_1)$ is the index of the trajectory of $\widetilde{z}$ along $\widetilde{I}_1$ relatively to $\widetilde{z}_1$, so it is equal to the index of δ relatively to $\widetilde{z}_1$. When $\widetilde{z}$ is in $\widetilde{W}_1$, the loop δ is included in $\widetilde{W}_1$ and is transverse to $\widetilde{\mathcal{F}}_1$, so $L(\widetilde{I}_1,\widetilde{z},\widetilde{z}_1)$ is positive. When $\widetilde{z}$ is not in $\widetilde{W}_1$, it is in one of the connected component of $\widetilde{M}_{Y_0} \setminus \overline{\widetilde{W}_1}$, and so is δ, therefore $L(\widetilde{I}_1,\widetilde{z},\widetilde{z}_1)$ is equal to 0. Since $\widetilde{I}^*$ fixes $\widetilde{z}_0$ and $\widetilde{z}_1$, the linking number $L(\widetilde{I}^*,\widetilde{z}_0,\widetilde{z}_1)$ is equal to 0. Referring to Section 2.15, we know that

$$L(\widetilde{I}_1,\widetilde{z}_0,\widetilde{z}_1) = L(\widetilde{I}^*,\widetilde{z}_0,\widetilde{z}_1) - 1 = -1,$$

and find a contradiction. □

The following lemma is a consequence of the previous one.

Lemma 4.16. *Let us suppose that (Y, I_Y) is maximal in $(\mathcal{I}, \precsim)$, that I_Y is not torsion-low at $z \in Y$, and that $M \setminus (Y \setminus \{z\})$ is neither a sphere nor a plane. If for every maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and every point $z' \in Y' \setminus (Y \setminus \{z\})$, $I_{Y'}$ is not torsion-low at z', then there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of $(Y \setminus \{z\}, I_Y)$ such that $\#(Y' \setminus (Y \setminus \{z\})) = \infty$.*

Proof. Fix a couple (Y, I_Y) maximal in $(\mathcal{I}, \precsim)$ and $z \in Y$ satisfying the assumptions of the lemma. By the previous lemma, there exists a maximal extension (Y_1, I_{Y_1}) of $(Y \setminus \{z\}, I_Y)$

such that $\#(Y_1 \setminus (Y \setminus \{z\})) > 1$. If $\#(Y_1 \setminus (Y \setminus \{z\})) = \infty$, the proof is finished; if $\#(Y_1 \setminus (Y \setminus \{z\}) < \infty$, we fix a point $z_1 \in Y_1 \setminus (Y \setminus \{z\})$. By hypothesis, I_{Y_1} is not torsion-low at z_1 and $M \setminus (Y_1 \setminus \{z_1\})$ is neither a sphere nor a plane. Since a maximal extension of $(Y_1 \setminus \{z_1\}, I_{Y_1})$ is also a maximal extension of $(Y \setminus \{z\}, I_Y)$, the couple (Y_1, I_{Y_1}) and $z_1 \in Y_1$ satisfies the assumptions of the previous lemma. We apply the previous lemma, and deduce that there exists a maximal extension $(Y_2, I_{Y_2}) \in \mathcal{I}$ of $(Y_1 \setminus \{z_1\}, I_{Y_1})$ such that $\#(Y_2 \setminus (Y_1 \setminus \{z_1\})) > 1$. If $\#(Y_2 \setminus (Y_1 \setminus \{z_1\})) = \infty$, the proof is finished; if $\#(Y_2 \setminus (Y_1 \setminus \{z_1\})) < \infty$, we continue the construction...

Then, either we end the proof in finitely many steps, or we can construct a strictly increasing sequence

$$(Y \setminus \{z\}, I_Y) \prec (Y_1 \setminus \{z_1\}, I_{Y_1}) \prec (Y_2 \setminus \{z_2\}, I_{Y_2}) \prec (Y_3 \setminus \{z_3\}, I_{Y_3}) \cdots$$

By Proposition 2.7, there exists an upper bound $(Y_\infty, I_{Y_\infty}) \in \mathcal{I}$ of this sequence, where $Y_\infty = \overline{\cup_{n\geq 1}(Y_n \setminus \{z_n\})}$. By Theorem 2.8, there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of (Y_∞, I_{Y_∞}). It is also a maximal extension of $(Y \setminus \{z\}, I_Y)$, and satisfies $\#(Y' \setminus (Y \setminus \{z\})) = \infty$. □

Proof of Proposition 4.11. We will prove this proposition by contradiction. Fix a maximal element $(Y, I_Y) \in \mathcal{I}$ and $z_0 \in Y$ such that I_Y is not torsion-low at z_0 and $M \setminus (Y \setminus \{z_0\})$ is neither a sphere nor a plane. Write $Y_0 = Y \setminus \{z_0\}$, and suppose that for all maximal extension $(Y', I_{Y'})$ of (Y_0, I_Y) and $z' \in Y' \setminus Y_0$, $I_{Y'}$ is not torsion-low at z'. By the previous lemma, there exists a maximal extension $(Y', I_{Y'})$ of (Y_0, I_Y) such that $\#(Y' \setminus Y_0) = \infty$.

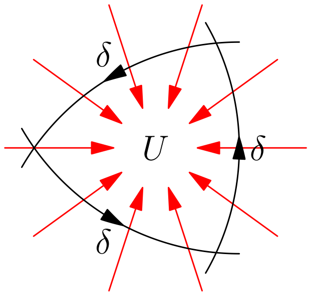
Let $\pi_{Y_0} : \widetilde{M}_{Y_0} \to M \setminus Y_0$ be the universal cover, $\widetilde{I}$ be the identity isotopy that lifts $I_Y|_{M\setminus Y_0}$, $\widetilde{I}'$ be the identity isotopy that lifts $I_{Y'}|_{M\setminus Y_0}$, and $\widetilde{f}$ be the lift of $f|_{M\setminus Y_0}$ associated to $I_Y|_{M\setminus Y_0}$. Since both I_Y and $I_{Y'}$ are maximal and $M \setminus Y_0$ is neither a sphere nor a plane, the point z_0 does not belong to Y'. Moreover, $\widetilde{f}$ is also the lift of $f|_{M\setminus Y_0}$ associated to $I_{Y'}|_{M\setminus Y_0}$. In particular, $\widetilde{f}$ fixes every point in $\pi_{Y_0}^{-1}(\{z_0\} \cup Y' \setminus Y_0)$. Fix $\widetilde{z}_0 \in \pi_{Y_0}^{-1}\{z_0\}$.

Sublemma 4.17. *For every $z \in Y' \setminus Y_0$, there exists $\widetilde{z} \in \pi_{Y_0}^{-1}\{z\}$ such that $\widetilde{z}_0$ and $\widetilde{z}$ are linked relatively to $\widetilde{I}$.*

Proof. Let $\mathcal{F}$ be a transverse foliation of I_Y, and $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}|_{M\setminus Y_0}$ to $\widetilde{M}_{Y_0}$. Fix $z \in Y' \setminus Y_0$ and $\widetilde{z} \in \pi_{Y_0}^{-1}\{z\}$. Since I_Y is a maximal identity isotopy, the trajectory of $\widetilde{z}$ along $\widetilde{I}$ is a loop that is not homotopic to zero in $\widetilde{M}_{Y_0} \setminus \pi_{Y_0}^{-1}\{z_0\}$. Let δ be a loop that is transverse to $\widetilde{\mathcal{F}}$, and is homotopic to the trajectory of $\widetilde{z}$ along $\widetilde{I}$ in $\widetilde{M}_{Y_0} \setminus \pi_{Y_0}^{-1}\{z_0\}$. By choosing suitable δ, we can suppose that δ intersects itself at most finitely many times, that each intersection point is a double point, and that the intersections are transverse. So, $\widetilde{M}_{Y_0} \setminus \delta$ has finitely many components, and we can define a locally constant function $\Lambda : \widetilde{M}_{Y_0} \setminus \delta \to \mathbb{Z}$ such that

- Λ is equal to 0 in the component of $\widetilde{M}_{Y_0} \setminus \delta$ that is not relatively compact;
- $\Lambda(\widetilde{z}') - \Lambda(\widetilde{z}'')$ is equal to the (algebraic) intersection number of δ and any arc from $\widetilde{z}''$ to $\widetilde{z}'$.

This function is not constant, and we have either $\max \Lambda > 0$ or $\min \Lambda < 0$. Suppose that we are in the first case (the other case can be treated similarly). Let U be a component of $\widetilde{M}_{Y_0} \setminus \delta$ such that Λ is equal to $\max \Lambda > 0$ in U. As in the picture, the boundary of U is a sub-curve of δ with the orientation such that U is to the left of its boundary, and is also transverse to $\widetilde{\mathcal{F}}$. So, there exists a singularity of $\widetilde{\mathcal{F}}$ in U. Note the fact that the set of singularities of $\widetilde{\mathcal{F}}$ is $\mathrm{Fix}(\widetilde{I}) = \pi_{Y_0}^{-1}\{z_0\}$. So, there exists an automorphism T of the universal cover space such that $T(\widetilde{z}_0)$ belongs to U, and the index of δ relatively to $T(\widetilde{z}_0)$



is positive. Note also that the linking number $L(\widetilde{I}, \widetilde{z}, T(\widetilde{z}_0))$ is equal to the index of δ relatively to $T(\widetilde{z}_0)$, and hence is equal to $\Lambda(T(\widetilde{z}_0))$ by definition of Λ. So, $T(\widetilde{z}_0)$ and $\widetilde{z}$ are linked relatively to $\widetilde{I}$. Consequently, $\widetilde{z}_0$ and $T^{-1}(\widetilde{z})$ are linked relatively to $\widetilde{I}$. $\square$

As in the proof of Lemma 4.15, we know that $\widetilde{f}$ can be blown-up at ∞. Since ∞ is accumulated by both the points in $\pi_{Y_0}^{-1}\{z_0\}$ and the points in $\pi_{Y_0}^{-1}(Y' \setminus Y_0)$, both $\rho_s(\widetilde{I}, \infty)$ and $\rho_s(\widetilde{I}', \infty)$ contain 0. Then, both $\rho_s(\widetilde{I}, \infty)$ and $\rho_s(\widetilde{I}', \infty)$ are reduced to 0, so $\widetilde{I}$ and $\widetilde{I}'$ are equivalent as local isotopies at ∞. Therefore, for every point $z \in Y' \setminus Y_0$, there exists $\widetilde{z} \in \pi_{Y_0}^{-1}\{z\}$ such that $\widetilde{z}_0$ and $\widetilde{z}$ are linked relatively to $\widetilde{I}'$. Let us denote by L the set of points $\widetilde{z} \in \pi_{Y_0}^{-1}(Y' \setminus Y_0)$ such that $\widetilde{z}$ and $\widetilde{z}_0$ are linked relatively to $\widetilde{I}'$. It contains infinitely many points.

Let γ be the trajectory of $\widetilde{z}_0$ along the isotopy $\widetilde{I}'$, and V be the connected component of $\widetilde{M}_{Y_0} \setminus \gamma$ containing ∞. Then $K = \widetilde{M}_{Y_0} \setminus V$ is a compact set that contains all the fixed points of $\widetilde{I}'$ that are linked with $\widetilde{z}_0$ relatively to $\widetilde{I}'$. In particular, $L \subset K$. Then, there exists $\widetilde{z}' \in K$ that is accumulated by points of L. We know that $\mathrm{Fix}(\widetilde{I}')$ is a closed set. So, $\widetilde{z}'$ belongs to $\mathrm{Fix}(\widetilde{I}') = \pi^{-1}(Y' \setminus Y_0)$. We find a point $\widetilde{z}'$ that is not isolated in $\pi_{Y_0}^{-1}(Y' \setminus Y_0)$, and a point $z' = \pi_Y(\widetilde{z}')$ that is not isolated in Y'. This means that $I_{Y'}$ is torsion-low at z'. We get a contradiction. $\square$

Proof of Proposition 4.12. We only need to prove that there exists $(X, I_X) \in \mathcal{I}_0$ such that $X \neq \emptyset$, because one knows $(\emptyset, I) \precsim (X, I_X)$ for all $(X, I_X) \in \mathcal{I}$ when M is a plane.

One has to consider the following two cases:

- Suppose that $\mathrm{Fix}(f)$ is reduced to one point z_0. In this case, similarly to the proof of Theorem 1.5, we can find an isotopy I_0 that fixes z_0 and is torsion-low at z_0. Then, $(\{z_0\}, I_0)$ belongs to $\mathcal{I}_0$.
- Suppose that $\mathrm{Fix}(f)$ contains at least two points. In this case, there exists a maximal $(Y, I_Y) \in \mathcal{I}$ such that $\#Y \geq 2$. If I_Y is torsion-low at a point in Y, the proof is finished; if I_Y is not torsion-low at every $z \in Y$, we fix $z_0 \in Y$ and can find a maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z_0\}, I_Y)$ and $z' \in Y' \setminus (Y \setminus \{z_0\})$ such that $I_{Y'}$ is torsion-low at z' by Proposition 4.11. Consequently, $(\{z'\}, I_{Y'})$ belongs to $\mathcal{I}_0$.

$\square$

Proof of Proposition 4.13. One knows $(X, I_X) \precsim (Y, I_Y)$ for all $(Y, I_Y) \in \mathcal{I}$ satisfying $X \subset Y$, when M is a sphere and $\#X \leq 1$. So, we only need to prove the following two facts:

i) there exists $(X, I_X) \in \mathcal{I}_0$ such that $X \neq \emptyset$;

ii) given $(X, I_X) \in \mathcal{I}_0$ such that $\#X = 1$, there exists $(X', I_{X'}) \in \mathcal{I}_0$ such that $X \subsetneqq X'$.

One has to consider the following two cases:

- Suppose that $\#\mathrm{Fix}(f) = 2$. In this case, we will prove that there exists an identity isotopy that fixes both fixed points and is torsion-low at each fixed point, which implies both i) and ii).

Denote by N and S the two fixed points. Since both N and S are isolated fixed points, we can find an identity isotopy I that fixes both N and S and is torsion-low at S. We will prove that I is also torsion-low at N.
Let J_N (resp. J_S) be an identity isotopy of the identity map of the sphere that fixes both N and S and satisfies $\rho_s(J_N, N) = \{1\}$ (resp. $\rho_s(J_S, S) = \{1\}$). One knows that the restrictions to $M \setminus \{N, S\}$ of J_N and J_S^{-1} are equivalent.
For every $k \geq 1$, since I is torsion-low at S, $J_S^{-k} I$ has a negative rotation type as a local isotopy at S. Let $\mathcal{F}_k$ be a transverse foliation of $J_S^{-k} I$. Then S is a source of $\mathcal{F}_k$. Since f is area preserving and $\mathcal{F}_k$ has exactly two singularities, N is a sink of $\mathcal{F}_k$. Note the fact that the restrictions to $M \setminus \{S, N\}$ of $J_N^k I$ and $J_S^{-k} I$ are homotopic. So, $J_N^k I$ has a positive rotation type as a local isotopy at N.
Similarly, for every $k \geq 1$, $J_N^{-k} I$ has a negative rotation type as a local isotopy at N. Therefore, I is torsion-low at N.

- Suppose that $\#\mathrm{Fix}(f) \geq 3$.
In this case, there exists $(Y, I_Y) \in \mathcal{I}$ such that $\#Y \geq 3$. We can prove i) by a similar discussion to the second part of the proof of Proposition 4.12. We can also give the following direct proof. Fix a maximal $(Y, I_Y) \in \mathcal{I}$ such that $\#Y \geq 3$. If Y is infinite, there exists a point $z \in Y$ that is not isolated in Y, and hence I_Y is torsion-low at z. If Y is finite, we consider a transverse foliation of I_Y and know that there is a saddle singulary point z of $\mathcal{F}$ by the Poincaré-Hopf formula and Remark 2.13, and hence I_Y is torsion-low at z. Anyway, there exists $z \in Y$ such that $(\{z\}, I_Y) \in \mathcal{I}_0$.
To prove ii), we fix $(X, I_X) \in \mathcal{I}_0$ such that $X = \{S\}$. For a maximal extension $(Y, I_Y) \in \mathcal{I}$ of (X, I_X) such that I_Y is torsion-low at S, one knows that $Y \setminus X$ is not empty. If I_Y is torsion-low at another fixed point, the proof is finished; if I_Y is not torsion-low at any other fixed points and if $\#(Y \setminus X)$ is bigger that 1, we get the result as a corollary of Proposition 4.11. Then, we only need to prove that there exists a maximal extension $(Y, I_Y) \in \mathcal{I}$ of (X, I_X) such that I_Y is torsion-low at S and that satisfies one of the two conditions: I_Y is torsion-low at another fixed point or $\#(Y \setminus X) > 1$.
Fix a maximal extension $(Y, I_Y) \in \mathcal{I}$ of (X, I_X) such that $\rho_s(I_Y, S) = \rho_s(I_X, S)$. Of course, I_Y is torsion-low at S. If I_Y is torsion-low at another fixed point or if $\#(Y \setminus X) > 1$, the proof is finished. Now, we suppose that $Y = \{S, N\}$ and I_Y is not torsion-low at N. One has to consider two cases: S is isolated in $\mathrm{Fix}(f)$ or not.
a) Suppose that S is isolated in $\mathrm{Fix}(f)$. As in the proof of Lemma 4.15, one has two consider the following four cases:
  - N is an isolated fixed point of f and there exists a local isotopy $I'_N > I_Y$ at N which does not have a positive rotation type;
  - N is not an isolated fixed point of f and $\rho_s(I_Y, N) \subset [-\infty, -1)$;
  - N is an isolated fixed point of f and there exists a local isotopy $I'_N < I_Y$ at N which does not have a negative rotation type;
  - N is not an isolated fixed point of f and $\rho_s(I_Y, N) \subset (1, +\infty]$.

  As before, we study the first two cases.
  Let $\mathcal{F}_Y$ be a transverse foliation of I_Y. As in the proof of Lemma 4.15, N is a source of $\mathcal{F}_Y$. Since f is area preserving and $\mathcal{F}_Y$ has exactly two singularities, S is a sink of $\mathcal{F}_Y$.
  Let I' be a maximal extension of $(Y, J_N I_Y)$. Since I_Y is torsion-low at S and I' is equivalent to $J_S^{-1} I_Y$ as local isotopies at S, I' has a negative rotation type at S. Moreover, as local isotopies at S, $J_S^k I' \sim J_S^{k-1} I_Y$ has a positive rotation type at S for $k \geq 1$, and has a negative rotation type for $k \leq -1$. Therefore I' is torsion-low

at S.
Let $\mathcal{F}'$ be a transverse foliation of I'. One knows that S is a source of $\mathcal{F}'$. Like in the proof of Lemma 4.15, we deduce that N is not a sink of $\mathcal{F}'$ by our assumption at the beginning of this case. Therefore, $\mathcal{F}'$ has another singularity, and hence one deduces that $\mathrm{Fix}(I') \geq 3$.

b) Suppose that S is not isolated in $\mathrm{Fix}(f)$. We know that $\rho_s(I_Y, S) \cap [-1,1] \neq \emptyset$ by definition.
We define the rotation number of a fixed point near S as in the proof of Proposition 1.10. By the maximality of I_Y, the rotation number of a fixed point near S is not zero. Then, either there exists $k \in \mathbb{Z} \setminus \{0\}$ such that S is accumulated by fixed points of f with rotation number k, or $\rho_s(I_Y, S)$ intersects $\{\pm\infty\}$. In the second case, the interior of the convex hull of $\rho_s(I_Y, S)$ contains a non-zero integer k', and hence 0 is in the interior of the convex hull of $\rho_s(J_S^{-k'} I_Y, S)$. So, S is accumulated by contractible fixed points of $J_S^{-k'} I_Y$ by the assertion iii) of Proposition 2.20, and hence is accumulated by fixed points with rotation k' (associate to I_Y). Anyway, there exists $k \in \mathbb{Z} \setminus \{0\}$ such that S is accumulated by fixed points of f with rotation number k. We fix one such k.
Let I' be a maximal extension of $J_S^{-k} I_Y$. Then, I' fixes at least 3 fixed points, and 0 belongs to $\rho_s(I', S)$. Therefore, I' is torsion-low at S, and satisfies $\#\mathrm{Fix}(I') \geq 3$. □

Remark 4.18. In both case a) and case b), we construct an identity isotopy I' that is torsion-low at S and has at least three fixed points. Even though $\rho_s(I_X, S)$ and $\rho_s(I', S)$ are different, I' is still an extension of (X, I_X) because M is a sphere and X is reduced to a single point. However, as was in Remark 4.6, for $(X', I_{X'}) \in \mathcal{I}_0$ that is a maximal extension of (X, I_X), $I_{X'}$ and I_X are not necessarily equivalent as local isotopies at S.

Now, let us prove Proposition 1.9.

Proof of Proposition 1.9. Let f be an area preserving homeomorphism of M that is isotopic to the identity and has finitely many fixed points. When $\mathrm{Fix}(f)$ is empty, the proposition is trivial. So, we suppose that $\mathrm{Fix}(f)$ is not empty. Let

$$n = \max\{\#\mathrm{Fix}(I) : I \text{ is an identity isotopy of } f\}.$$

One has to consider the following three cases:

- Suppose that M is a plane and f has exactly one fixed point. As in the first part of the proof of Proposition 4.12, there exists an identity isotopy that fixes this fixed point and is torsion-low at this fixed point.
- Suppose that M is a sphere and f has exactly two fixed points. As in the first part of the proof of Proposition 4.13, there exists an identity isotopy that fixes these two fixed points and is torsion-low at each fixed point.
- Suppose that we are not in the previous two cases. Let $\mathfrak{I}$ be the set of identity isotopies of f with n fixed points. It is not empty. We can give a preorder $\lhd$ over $\mathfrak{I}$ such that $I \lhd I'$ if and only if

$$\#\{z \in \mathrm{Fix}(I), I \text{ is torsion-low at } z\} \leq \#\{z \in \mathrm{Fix}(I'), I' \text{ is torsion-low at } z\}.$$

Since $\#\{z \in \mathrm{Fix}(I), I \text{ is torsion-low at } z\}$ is not bigger than n for all $I \in \mathfrak{I}$, $\mathfrak{I}$ has a maximal element. Fix a maximal element I of $\mathfrak{I}$. We will prove by contradiction that I is torsion-low at every $z \in \mathrm{Fix}(I)$.

Suppose that I is not torsion-low at $z_0 \in \mathrm{Fix}(I)$. Write $Y_0 = \mathrm{Fix}(I) \setminus \{z_0\}$. Since we are not in the previous two cases, $M \setminus Y_0$ is neither a plane nor a sphere. By Proposition 4.11, there exist a maximal extension I' of (Y_0, I) and $z' \in \mathrm{Fix}(I') \setminus Y_0$ such that I' is torsion-low at z'. This contradicts the fact that I is maximal in $(\mathfrak{J}, \lhd)$. □

4.2.2 Proof of Theorem 4.5 when Fix(f) is not totally disconnected

We suppose that Fix(f) is not totally disconnected in this subsection. In this case, the proof of Theorem 4.5 is similar to the one in the previous section except that we should consider more cases. More precisely, Theorem 4.5 is a corollary of Zorn's lemma and the following four similar propositions. The proof of Proposition 4.19 is just a copy of the one of Proposition 4.10; while the proofs of the others are the aim of this subsection.

Proposition 4.19. *If $\{(X_\alpha, I_{X_\alpha})\}_{\alpha\in J}$ is a totally ordered chain in $\mathcal{I}_0$, then there exists an upper bound $(X_\infty, I_{X_\infty}) \in \mathcal{I}_0$ of the chain, where $X_\infty = \overline{\cup_{\alpha\in J} X_\alpha}$*

Proposition 4.20. *For every maximal $(Y, I_Y) \in \mathcal{I}$ and $z \in Y$ such that I_Y is not torsion-low at z and $M \setminus (Y \setminus \{z\})$ is neither a sphere nor a plane[1] whose boundary is empty or reduced to one point, there exist a maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and $z' \in Y' \setminus (Y \setminus \{z\})$ such that $I_{Y'}$ is torsion-low at z'.*

Proposition 4.21. *When M is a plane, $(X, I_X) \in \mathcal{I}_0$ is not maximal in $(\mathcal{I}_0, \precsim)$ if $X = \emptyset$.*

Proposition 4.22. *When M is a sphere, $(X, I_X) \in \mathcal{I}_0$ is not maximal in $(\mathcal{I}_0, \precsim)$ if $\#X \leq 1$.*

To prove Proposition 4.20, we need the following Lemmas 4.23-4.25. Lemma 4.23 is almost the same as Lemma 4.15 except that we deal with the the connected component of $M \setminus (Y \setminus \{z\})$ containing z instead of $M \setminus (Y \setminus \{z\})$. Similarly to the proof of Lemma 4.16, we will get Lemma 4.25 by Lemma 4.23 and Lemma 4.24. Then, as in the proof of Proposition 4.11, we can give a similar proof of Proposition 4.20 as a corollary of Lemma 4.25. The new case is Lemma 4.24.

Lemma 4.23. *Suppose that (Y, I_Y) is maximal in $\mathcal{I}$, that I_Y is not torsion-low at $z \in Y$, and that the connected component of $M \setminus (Y \setminus \{z\})$ containing z is neither a sphere nor a plane. If for every maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and every point $z' \in Y' \setminus (Y \setminus \{z\})$, $I_{Y'}$ is not torsion-low at z', then there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of $(Y \setminus \{z\}, I_Y)$ such that $\#(Y' \setminus (Y \setminus \{z\})) > 1$.*

Proof. The proof of Lemma 4.23 is just a copy of the one of Lemma 4.15 except that we should replace $M \setminus Y_0$ with the the connected component of $M \setminus Y_0$ containing z_0. □

Lemma 4.24. *Suppose that (Y, I_Y) is maximal in $\mathcal{I}$, that I_Y is not torsion-low at $z \in Y$, and that the connected component of $M \setminus (Y \setminus \{z\})$ containing z is a plane whose boundary in M contains more that two points of $Y \setminus \{z\}$. If for every maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and every $z' \in Y' \setminus (Y \setminus \{z\})$, $I_{Y'}$ is not torsion-low at z', then there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of $(Y \setminus \{z\}, I_Y)$ such that $\#(Y' \setminus (Y \setminus \{z\})) > 1$.*

Proof. Fix a maximal $(Y, I_Y) \in \mathcal{I}$ and $z_0 \in Y$ satisfying the assumptions of this lemma. Write $Y_0 = Y \setminus \{z_0\}$, and denote by M_{Y_0} the connected component of $M \setminus Y_0$ containing z_0. Then M_{Y_0} is a plane and $\#\partial M_{Y_0} > 1$. As in the proof of Lemma 4.15, since I_Y is not torsion-low at z_0, one has to consider the following four cases:

1. Here, a plane means an open set that is homeomophic to $\mathbb{R}^2$

- z_0 is an isolated fixed point of f and there exists a local isotopy $I'_{z_0} > I_Y$ at z_0 which does not have a positive rotation type;
- z_0 is not an isolated fixed point of f and $\rho_s(I_Y, z_0) \subset [-\infty, -1)$;
- z_0 is an isolated fixed point of f and there exists a local isotopy $I'_{z_0} < I_Y$ at z_0 which does not have a negative rotation type;
- z_0 is not an isolated fixed point of f and $\rho_s(I_Y, z_0) \subset (1, +\infty]$.

As before, we only study the first two cases.

Let $\mathcal{F}_Y$ be a transverse foliation of I_Y. As in the proof of Lemma 4.15, we know that z_0 is a source of $\mathcal{F}_Y$.

Since $\#\partial M_{Y_0} > 1$, the plane M_{Y_0} can be blown-up by prime-ends at infinity. Because I_Y fixes ∂M_{Y_0} and z_0, $I_Y|_{M_{Y_0}}$ can be viewed as a local isotopy at ∞, and the blow-up rotation number $\rho(I_Y|_{M_{Y_0}}, \infty)$, that was defined in Section 2.10, is equal to 0.

Let I^* be a maximal extension of $(\{z_0\}, J_{z_0} I_Y|_{M_{Y_0}})$, and $\mathcal{F}^*$ be a transverse foliation of I^*. Note that I_Y is not torsion-low at z_0, by the same argument of the proof of Lemma 4.15, we know that z_0 is not a sink of $\mathcal{F}^*$.

We can assert that ∞ is a source of $\mathcal{F}^*$. Indeed, when the total area of M_{Y_0} is finite, $f|_{M_{Y_0}}$ is area preserving as a local homeomorphism at ∞, so $\rho_s(I_Y|_{M_{Y_0}}, \infty)$ is not empty by Proposition 2.18 and is reduced to 0 by the assertion vii) of Proposition 2.20. Then, by the assertion i) of Proposition 2.20, $\rho_s(I^*, \infty)$ is reduced to -1, and by the assertion v) of Proposition 2.20, ∞ is a source of $\mathcal{F}^*$. However, the total area of M_{Y_0} may be infinite. In this case, we can not get the result that $\rho_s(I_Y|_{M_{Y_0}}, \infty)$ is not empty. But anyway, we can prove the assertion by considering the following two cases:

- Suppose that $\rho_s(I_Y|_{M_{Y_0}}, \infty)$ is not empty. As in the case where the total area of M_{Y_0} is finite, $\rho_s(I_Y|_{M_{Y_0}}, \infty)$ is reduced to 0, and $\rho_s(I^*, \infty)$ is reduced to -1. Therefore, ∞ is a source of $\mathcal{F}^*$ by the assertion v) of Proposition 2.20.
- Suppose that $\rho_s(I_Y|_{M_{Y_0}}, \infty)$ is empty. Since $f|_{M_{Y_0}}$ is area preserving, $f|_{M_{Y_0}}$ is not conjugate to a contraction or an expansion at ∞. By Proposition 2.18, the germ of $f|_{M_{Y_0}}$ at ∞ is conjugate to a local homeomorphism $z \mapsto e^{i2\pi\frac{p}{q}} z(1 + z^{qr})$ at 0 with $q, r \in \mathbb{N}$ and $p \in \mathbb{Z}$. Since $\rho(I_Y|_{M_{Y_0}}, \infty) = 0$, we can deduce that $p \in q\mathbb{Z}$. Therefore, one has $i(f|_{M_{Y_0}}, \infty) > 1$. Let $I_0 = (g_t)_{t\in[0,1]}$ be a local isotopy at 0 such that $g_t(z) = z(1 + tz^{qr})$. Then, $\rho(I_0, 0)$ is equal to 0 and $i(I_0, 0)$ is positive. Since $\rho(I_Y|_{M_{Y_0}}, \infty)$ is equal to 0, $I_Y|_{M_{Y_0}}$ is conjugate to a local isotopy that is in the same homotopy class of I_0. So, $i(I_Y|_{M_{Y_0}}, \infty) = i(I_0, 0)$ is positive. Therefore, ∞ is a source of $\mathcal{F}^*$ by Proposition 2.14.

Then, like in the proof of Lemma 4.15, we deduce that I^* fixes finitely many points, that there exists a sink z_1 of $\mathcal{F}^*$, and that there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of (Y_0, I_Y) such that $Y_0 \cup \{z_1\} \subset Y'$ and $\#(Y' \setminus Y_0) > 1$. □

Lemma 4.25. *Suppose that (Y, I_Y) is maximal in $\mathcal{I}$, that I_Y is not torsion-low at $z \in Y$, and that the connected component of $M \setminus (Y \setminus \{z\})$ is neither a sphere nor a plane whose boundary is empty or reduced to one point. If for every maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and every $z' \in Y' \setminus (Y \setminus \{z\})$, $I_{Y'}$ is not torsion-low at z', then there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of $(Y \setminus \{z\}, I_Y)$ such that $\#(Y' \setminus (Y \setminus \{z\})) = \infty$.*

Proof. The proof is almost the same as the one of Lemma 4.16 except the following: every time we want to get a new couple, we should check that the previous couple satisfies the assumptions of Lemma 4.23 or Lemma 4.24 instead of the assumptions of Lemma 4.15. □

Now, we begin the proof of Proposition 4.20. The proof is similar to the one of Proposition 4.11.

Proof of Proposition 4.20. We will prove this proposition by contradiction. Fix a maximal $(Y, I_Y) \in \mathcal{I}$ and $z_0 \in Y$ such that I_Y is not torsion-low at z_0 and $M \setminus (Y \setminus z_0)$ is neither a sphere nor a plane whose boundary is empty or reduced to a single point. Write $Y_0 = Y \setminus \{z_0\}$, and denote by M_{Y_0} the connected component of $M \setminus Y_0$ containing z_0. Suppose that for all maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and $z' \in Y' \setminus (Y \setminus \{z\})$, $I_{Y'}$ is not torsion-low at z'. By the previous lemma, there exists a maximal extension $(Y', I_{Y'})$ of (Y_0, I_Y) such that $\#(Y' \setminus Y_0) = \infty$.

Let us prove by contradiction that $Y' \setminus Y_0 \subset M_{Y_0}$. Suppose that there exists $z_1 \in Y' \setminus Y_0$ that is in another component of $M \setminus Y_0$. Since $I_Y|_{M\setminus Y_0}$ and $I_{Y'}|_{M\setminus Y_0}$ are homotopic, the trajectory of z_1 along I_Y is homotopic to zero in $M \setminus Y_0$. Moreover, because the trajectory of z_1 along I_Y is in another component of $M \setminus Y_0$, this trajectory is homotopic to zero in $M \setminus Y$, which contradicts the maximality of (Y, I_Y) by Proposition 2.6.

Then, one has to consider two cases:

- M_{Y_0} is neither a sphere nor a plane,
- M_{Y_0} is a plane whose boundary contains more than two points.

In the first case, we repeat the proof of Proposition 4.11 except that we should replace $M \setminus Y_0$ with M_{Y_0}. In the second case, the idea is similar, but we do not lift the isotopies to the universal cover because M_{Y_0} itself is a plane. □

Proof of Proposition 4.21. As in the proof of Proposition 4.12, we only need to prove that there exists $(X, I_X) \in \mathcal{I}_0$ such that $X \neq \emptyset$.

Since $\mathrm{Fix}(f)$ is not totally disconnected, we can fix a connected component X of $\mathrm{Fix}(f)$ that is not reduced to a point. By Proposition 2.11, there exists a maximal identity isotopy I of f that fixes all the points in X. So, 0 belongs to $\rho_s(I, z)$ for all $z \in X$, and hence (X, I) belongs to $\mathcal{I}_0$. □

Proof of Proposition 4.22. As in the proof of Proposition 4.13, we only need to prove the following two facts:

i) there exists $(X, I_X) \in \mathcal{I}_0$ such that $X \neq \emptyset$;

ii) given $(X, I_X) \in \mathcal{I}_0$ such that $\#X = 1$, there exists $(X', I_{X'}) \in \mathcal{I}_0$ such that $X \subsetneq X'$.

The proof of the first fact is the same to the proof of Proposition 4.21; while the proof of the second fact is similar to the proof of Proposition 4.13 in the case $\#\mathrm{Fix}(f) \geq 3$. □

4.3 Examples

In this section, we will give some explicit examples to get the optimality of previous results.

Exemple 4.26. (A local isotopy that has both positive and negative rotation types)

Write f_t for the homothety of factor $1 + t$ of a plane. One can note that f_1 has an isolated fixed point 0, and $I = (f_t)_{t\in[0,1]}$ has both positive and negative rotation types at 0. In fact, let

$$\begin{array}{rcl} \pi : \mathbb{R} \times (-\infty, 0) & \to & \mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\} \\ (\theta, y) & \mapsto & -ye^{i2\pi\theta} \end{array}$$

be the universal cover. Let $\mathcal{F}'_1$ be the foliation on $\mathbb{R} \times (-\infty, 0)$ whose leaves are the lines $y = \theta + c$ upward. It descends to an oriented foliation $\mathcal{F}_1$ on $\mathbb{C} \setminus \{0\}$ that is locally transverse to I. Moreover, 0 is a sink of $\mathcal{F}_1$. Let $\mathcal{F}'_2$ be the foliation on $\mathbb{R} \times (-\infty, 0)$ whose leaves are the lines $y = -\theta + c$ downward. It descends to an oriented foliation $\mathcal{F}_2$ on $\mathbb{C} \setminus \{0\}$ that is locally transverse to I. Moreover, 0 is a source of $\mathcal{F}_2$.

(a) $\mathcal{F}_1$ (b) $\mathcal{F}_2$

Figure 4.1: The two foliations of Example 4.26

Exemple 4.27. (A local isotopy that has both positive and zero rotation types)

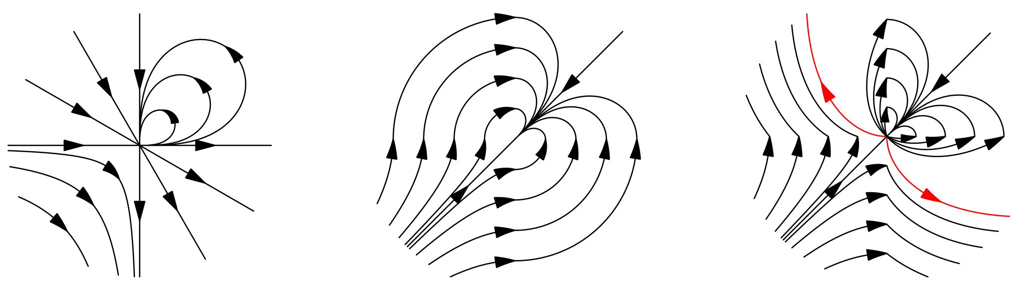
We define a flow[2] on $\mathbb{R}^2$ by

$$f_t(x,y) = \begin{cases} \frac{x^2+y^2}{x^2e^{-2t}+y^2e^{2t}}(xe^{-t}, ye^{t}) & \text{for } x \geq 0, y \geq 0, \\ (xe^{-t}, ye^{-t}) & \text{for } x \leq 0, y \geq 0, \\ (xe^{-t}, ye^{t}) & \text{for } x \leq 0, y \leq 0, \\ (xe^{t}, ye^{t}) & \text{for } x \geq 0, y \leq 0. \end{cases}$$

It is the flow of the (time-independent) continuous vector field V in the plane $\mathbb{R}^2$, where V is defined by

$$V(x,y) = \begin{cases} (\frac{x(x^2-3y^2)}{x^2+y^2}, \frac{y(3x^2-y^2)}{x^2+y^2}) & \text{for } x > 0, y > 0, \\ (-x, -y) & \text{for } x \leq 0, y \geq 0, \\ (-x, y) & \text{for } x \leq 0, y \leq 0, \\ (x, y) & \text{for } x \geq 0, y \leq 0. \end{cases}$$

Then, $f = f_1$ has a unique fixed point 0, and $I = (f_t)_{t\in[0,1]}$ is an identity isotopy of f. We will prove that I has both positive and zero rotation types at 0 by constructing two transverse foliations $\mathcal{F}_1$, $\mathcal{F}_2$ of I such that 0 is a sink of $\mathcal{F}_1$ and is a mixed singularity of $\mathcal{F}_2$.



(a) The dynamics of the flow (b) $\mathcal{F}_1$ (c) $\mathcal{F}_2$

Figure 4.2: The dynamics and two foliations of Example 4.27

We will construct $\mathcal{F}_1$ by considering the integral curves of vector field. We define a

2. The flow on the first quadrant is just $f_t(z) = \frac{1}{\varphi_t(1/z)}$, where $z = x + iy$ is the complex coordinate and φ is the flow defined by $\varphi_t(x,y) = (xe^{-t}, ye^{t})$.

continuous vector field ξ in the plane by

$$\xi(x,y)=\begin{cases}(-\frac{y(3x^2-y^2)}{x^2+y^2},\frac{x(x^2-3y^2)}{x^2+y^2}) & \text{for } x>0, y>0,\\ (y,-x) & \text{for } x\leq 0, y\geq 0,\\ (-y,-x) & \text{for } x\leq 0, y\leq 0,\\ (-y,x) & \text{for } x\geq 0, y\leq 0.\end{cases}$$

One knows that ξ vanish at a unique point 0, is transverse to V, and satisfies $\det(V(x,y),\xi(x,y))>0$ for all $(x,y)\neq 0$. So, the the foliation $\mathcal{F}_1$ whose leaves are the integral curves of ξ is transverse to I. Moreover, by a direct computation, we can get the formulae of the integral curves of ξ and find that every integral curves go to 0 as in the picture. So, 0 is a sink of $\mathcal{F}_1$.

Let

$$\begin{array}{rcl}\pi:\mathbb{R}\times(-\infty,0) & \to & \mathbb{C}^2\setminus\{0\}\simeq\mathbb{R}^2\setminus\{0\}\\ (x,y) & \mapsto & -ye^{i2\pi x}\end{array}$$

be the universal cover, and $(\widetilde{f}_t)_{t\in[0,1]}$ be the identity isotopy that lifts I. We know that $\gamma_{(x,y)}: t\mapsto \widetilde{f}_t(x,y)$ is a vertical segment upward for every $(x,y)\in[\frac{1}{4},\frac{1}{2}]\times(-\infty,0)$, and is a vertical segment downward for every $(x,y)\in[-\frac{1}{4},0]\times(-\infty,0)$. We define an oriented foliation $\widetilde{\mathcal{F}}_{II}$ on the domain $(\frac{1}{4},\frac{1}{2})\times(-\infty,0)$ whose leaves are the the restriction to $(\frac{1}{4},\frac{1}{2})\times(-\infty,0)$ of the family of curves $(\ell_c)_{c\in(-1,\infty)}$ such that

- ℓ_c is the graph of $y=\log(4x-1-c)$ with the direction from right to left, for $c\in(-1,0]$,
- ℓ_c is the graph of $y=\log(4x-1)-c$ with the direction from right to left, for $c\in(0,\infty)$.

Then, $\gamma_{(x,y)}$ is positively transverse to $\widetilde{\mathcal{F}}_{II}$ for every $(x,y)\in(\frac{1}{4},\frac{1}{2})\times(-\infty,0)$. Similarly, we define an oriented foliation $\widetilde{\mathcal{F}}_{IV}$ on the domain $(-\frac{1}{4},0)\times(-\infty,0)$ whose leaves are the restriction to $(-\frac{1}{4},0)\times(-\infty,0)$ of the family of curves $(\ell'_c)_{c\in(-1,\infty)}$ such that

- ℓ'_c is the graph of $y=\log(-4x-c)$ with the direction from left to right, for $c\in(-1,0]$,
- ℓ'_c is the graph of $y=\log(-4x)-c$ with the direction from left to right, for $c\in(0,\infty)$.

Then, $\gamma_{(x,y)}$ is positively transverse to $\widetilde{\mathcal{F}}_{IV}$ for every $(x,y)\in(-\frac{1}{4},0)\times(-\infty,0)$.

Note the following facts:

- ℓ_c intersects $\{\frac{1}{4}\}\times(-\infty,0)$, and does not intersect $\{\frac{1}{2}\}\times(-\infty,0)$, for $c\in(-1,0)$,
- ℓ_0 intersects neither $\{\frac{1}{4}\}\times(-\infty,0)$ nor $\{\frac{1}{2}\}\times(-\infty,0)$,
- ℓ_c intersects $\{\frac{1}{2}\}\times(-\infty,0)$, and does not intersect $\{\frac{1}{4}\}\times(-\infty,0)$, for $c\in(0,\infty)$,
- ℓ'_c intersects $\{0\}\times(-\infty,0)$, and does not intersect $\{-\frac{1}{4}\}\times(-\infty,0)$, for $c\in(-1,0)$,
- ℓ'_0 intersects neither $\{0\}\times(-\infty,0)$ nor $\{-\frac{1}{4}\}\times(-\infty,0)$,
- ℓ'_c intersects $\{-\frac{1}{4}\}\times(-\infty,0)$, and does not intersect $\{0\}\times(-\infty,0)$, for $c\in(0,\infty)$.

We can define a transverse foliation $\mathcal{F}_2$ of I such that

- the restriction of $\mathcal{F}_2$ to the second quadrant II is equal to $\pi\circ\widetilde{\mathcal{F}}_{II}$,
- the restriction of $\mathcal{F}_2$ to the fourth quadrant IV is equal to $\pi\circ\widetilde{\mathcal{F}}_{IV}$,
- the restriction of $\mathcal{F}_2$ to $\mathbb{R}^2\setminus(II\cup IV)$ is equal to the restriction of $\mathcal{F}_1$ to the same set.

Moreover, one can deduce that 0 is a mixed singularity of $\mathcal{F}_2$.

Exemple 4.28. (An orientation and area preserving local homeomorphism whose local rotation set is reduced to ∞)

Let f be a homeomorphism of $\mathbb{C}$ defined by

$$f(z) = \begin{cases} 0 & \text{for } z = 0, \\ ze^{i2\pi/|z|} & \text{for } z \neq 0. \end{cases}$$

It is area preserving and fix 0. Moreover, $\rho_s(I, 0)$ is reduced to ∞ for every isotopy I of f fixing 0.

Exemple 4.29. (Example of Remark 1.8)

We will construct an orientation preserving diffeomorphism f of the sphere with 2 fixed points such that f is area preserving in a neighborhood of each fixed point but there does not exist any torsion-low maximal identity isotopy of f.

Let φ be a diffeomorphism of $[0, 1]$ that satisfies

$$\begin{cases} \varphi(y) = y & \text{for } y \in [0, 1/6] \cup [5/6, 1], \\ \varphi(y) < y & \text{for } y \in (1/6, 5/6). \end{cases}$$

Let g be a diffeomorphism of $\mathbb{R} \times [0, 1]$ that is defined by

$$g(x, y) = (x + 3y, \varphi(y)).$$

We define an equivalent relation $\sim$ on $\mathbb{R} \times [0, 1]$ such that

$$\begin{cases} (x, y) \sim (x + 1, y) & \text{for all } (x, y) \in \mathbb{R} \times (0, 1) \\ (x, 0) \sim (x', 0) & \text{for all } x, x' \in \mathbb{R} \\ (x, 1) \sim (x', 1) & \text{for all } x, x' \in \mathbb{R}. \end{cases}$$

Then, $\mathbb{R} \times [0, 1]/_\sim$ is a sphere, and g descends to a diffeomorphism f of the sphere that has two fixed points and is area preserving near each fixed point. Note the facts that every maximal identity isotopy I fixes both fixed points of f, that the rotation number of I at each fixed point is an integer, and that the sum of the rotation numbers of I at both fixed point is 3. By Proposition 1.6, there does not exist any torsion-low maximal identity isotopy of f.

Exemple 4.30. (Example of Remark 4.6)

In this example, we will construct an isotopy I^* on the sphere such that I^* is torsion-low at a fixed point z, but there does not exist any torsion-low maximal isotopy that is equivalent to I^* as a local isotopy at z.

We will induce the isotopy by generating functions (see Appendix 4.4).

Let φ be a smooth 1-periodic function on $\mathbb{R}$ that satisfies

$$\varphi(0) = \varphi(3/4) = \varphi(1) = 0 \text{ and } |\varphi| \leq \frac{1}{2\pi},$$

$$\begin{cases} \varphi(s) > 0 \text{ for } 0 < s < 3/4 \\ \varphi(s) < 0 \text{ for } 3/4 < s < 1 \end{cases}, \quad \text{and} \quad \int_0^1 \varphi(s) ds = 0,$$

$$|\varphi(s)| < s \sin^2 \frac{\pi}{s} \quad \text{for} \quad 3/4 < s < 1.$$

Let

$$g(x, y) = \begin{cases} 0 & \text{for } y \leq 0, \\ \int_0^y s \sin^2 \frac{\pi}{s} + \varphi(s) \sin^2 \pi x ds & \text{for } 0 < y < 1, \\ \int_0^1 s \sin^2 \frac{\pi}{s} ds & \text{for } y \geq 1. \end{cases}$$

Then, g is constant on $\mathbb{R}\times(-\infty,0]$ and on $\mathbb{R}\times[1,\infty)$ respectively, and satisfies $g(x+1,y)=g(x,y)$. Moreover, one knows

$$\partial^2_{12}g(x,y)=\begin{cases}0 & \text{for } y\leq 0 \text{ or } y\geq 1,\\ \pi\varphi(y)\sin(2\pi x) & \text{for } 0<y<1.\end{cases}$$

So, $\partial^2_{12}g\leq\frac{1}{2}<1$. Therefore, g defines an identity isotopy $I=(f_t)_{t\in[0,1]}$ by the following equations:

$$f_t(x,y)=(X^t,Y^t)\Leftrightarrow\begin{cases}X^t-x= & t\partial_2 g(X^t,y),\\ Y^t-y= & -t\partial_1 g(X^t,y),\end{cases}$$

For every $t\in[0,1]$, f_t is the identity on $\mathbb{R}\times(-\infty,0]\cup\mathbb{R}\times[1,\infty)$, and satisfies $f_t(x+1,y)=f_t(x,y)$. Moreover, for every $t\in(0,1]$, a point (x,y) is a fixed point of f_t if and only if it is a critical point of g. Let $\mathcal{F}$ be the foliation whose leaves are the integral curves of the gradient vector field $(x,y)\mapsto(\partial_1 g(x,y),\partial_2 g(x,y))$ of g. As will be proved in Appendix 4.4, $\mathcal{F}$ is a transverse foliation of I.

We know that

$$\partial_1 g(x,y)=\begin{cases}0 & \text{for } y\leq 0 \text{ or } y\geq 1,\\ \pi\sin(2\pi x)\int_0^y\varphi(s)ds & \text{for } 0<y<1,\end{cases}$$

and that

$$\partial_2 g(x,y)=\begin{cases}0 & \text{for } y\leq 0 \text{ or } y\geq 1,\\ y\sin^2\frac{\pi}{y}+\varphi(y)\sin^2(\pi x) & \text{for } 0<y<1.\end{cases}$$

So, the set of critical points of g is

$$C=\{(n,\frac{1}{m}):n\in\mathbb{Z},m\in\mathbb{N}\}\cup\mathbb{R}\times(-\infty,0]\cup\mathbb{R}\times[1,\infty),$$

and one deduces that $\partial_2 g(x,y)>0$ for $(x,y)\notin C$.

We define an equivalent relation $\sim$ on $\mathbb{R}^2$ by

$$\begin{cases}(x,y)\sim(x',y') & \text{for } y,y'\leq 0,\\ (x,y)\sim(x+1,y) & \text{for } 0<y<1,\\ (x,y)\sim(x',y') & \text{for } y,y'\geq 1.\end{cases}$$

Then, $\mathbb{R}^2/_\sim$ is a sphere, f_1 descends to an area preserving homeomorphism f' of the sphere, I descends to an identity isotopy I' of f', and $\mathcal{F}$ descends to a transverse foliation $\mathcal{F}'$ of I'. Moreover, one knows that $\mathrm{Fix}(I')=\mathrm{Fix}(f')=\mathrm{Sing}(\mathcal{F}')$, where $\mathrm{Sing}(\mathcal{F}')$ is the set of singularities of $\mathcal{F}'$. We denote by S and N the two points $\mathbb{R}\times(-\infty,0]$ and $\mathbb{R}\times[1,\infty)$ in the sphere respectively.

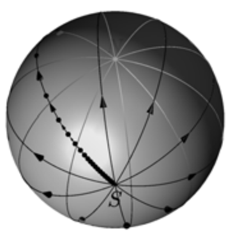


Figure 4.3: A sketch map of $\mathcal{F}'$

The fixed point S is not isolated in $\mathrm{Fix}(I')$, and so $\rho_s(I', S)$ is reduced to 0; N is isolated in $\mathrm{Fix}(f')$ and is a sink of $\mathcal{F}'$; and all the other fixed points of f' are isolated in $\mathrm{Fix}(f')$ and are saddles of $\mathcal{F}'$. Let I^* be an identity isotopy of f' fixing S such that $\rho_s(I^*, S)$ is reduced to -1. Then, I^* is torsion-low at S. We will prove that there does not exist any torsion-low maximal isotopy I'' such that $\rho_s(I'', S)$ is reduced to -1.

Indeed, a maximal identity isotopy of f' fixes either all the fixed points of f' (in which case, the isotopy is homotopic to I' relatively to $\mathrm{Fix}(f')$) or exactly two fixed points. If I'' is a maximal identity isotopy of f such that $\rho_s(I'', S)$ is reduced to -1, then I'' fixes exactly two fixed points. Denote by $\{S, z_1\}$ the set of fixed points of I''. One knows that z_1 is an isolated fixed point of f', and that $J_{z_1}^{-1} I''$ is equivalent to I' as local isotopies at z_1. Therefore, $J_{z_1}^{-1} I''$ does not have a negative rotation type at z_1, and hence I'' is not torsion-low at z_1.

Exemple 4.31. (Example of Remark 4.7)

In this example, we will construct an orientation and area preserving homeomorphism f of the sphere such that there does not exist any maximal identity isotopy I of f such that $0 \in \rho_s(I, z)$ for every $z \in \mathrm{Fix}(I)$ that is not an isolated fixed point of f.

Let g be a homeomorphism on $\mathbb{R} \times [0, 1]$ that is defined by

$$g(x, y) = \begin{cases} (x, y) & \text{for } 0 \le y \le \frac{1}{3}, \\ (x + 3y - 1, y) & \text{for } \frac{1}{3} < y \le \frac{2}{3}, \\ (x + 1, y) & \text{for } \frac{2}{3} < y \le 1. \end{cases}$$

We define an equivalent relation $\sim$ on $\mathbb{R} \times [0, 1]$ such that

$$\begin{cases} (x, 0) \sim (x', 0) & \text{for } x, x' \in \mathbb{R} \\ (x, y) \sim (x + 1, y) & \text{for } 0 < y < 1, \\ (x, 1) \sim (x', 1) & \text{for } x, x' \in \mathbb{R}. \end{cases}$$

Then, $\mathbb{R} \times [0, 1]/_\sim$ is a sphere, g descends to an orientation and area preserving diffeomorphism f of the sphere that has infinitely many fixed points, and every fixed point of f is not isolated in $\mathrm{Fix}(f)$. We will prove that there does not exist any maximal isotopy I such that for all $z \in \mathrm{Fix}(I)$, one has $0 \in \rho_s(I, z)$.

By definition of f, one knows that f can be blown-up at each fixed point, and hence for every identity isotopy I of f and every $z \in \mathrm{Fix}(I)$, the rotation set $\rho_s(I, z)$ is reduced to $\rho(I, z)$. Then, we only need to prove that there does not exist any maximal identity isotopy I such that $\rho(I, z) = 0$ for every $z \in \mathrm{Fix}(I)$.

Denote by N and S the two components of $\mathrm{Fix}(f)$ respectively. Note the following fact: for $z_1, z_2 \in \mathrm{Fix}(f)$ and every identity isotopy I fixing both z_1 and z_2, one can deduce

$$\rho(I, z_1) + \rho(I, z_2) = \begin{cases} 0 & \text{if } z_1, z_2 \in S, \text{ or if } z_1, z_2 \in N, \\ 1 & \text{if } z_1 \in S, z_2 \in N, \text{ or if } z_1 \in N, z_2 \in S. \end{cases}$$

Let us conclude the proof by observing the properties of any maximal identity isotopy of f. Indeed, if I is a maximal identity isotopy of f, it satisfies one of the following properties:

- The set of fixed points of I is the union of N (resp. S) and a point z in S (resp. N). In this case, $\rho(I, z) = 1$.
- The set of fixed points of I is the union of a point z_1 in N (resp. S) and a point z_2 in S (resp. N), and the rotation numbers satisfy $\rho(I, z_i) \neq 0$ for $i = 1, 2$.

- The set of fixed points of I is a subset of N (resp. S) with exactly two points z_1 and z_2, and the rotation numbers satisfies

$$\rho(I, z_1) = -\rho(I, z_2) \in \mathbb{Z} \setminus \{0\}.$$

Exemple 4.32. (Example of Remark 4.9)

We will construct an orientation and area preserving diffeomorphism of the sphere such that there does not exist any maximal identity isotopy I satisfying

$$-1 < \rho(I, z) < 1, \text{ for every } z \in \mathrm{Fix}(F).$$

Let g be a diffeomorphism of $\mathbb{R} \times [0, 1]$ that is defined by

$$g(x, y) = (x + y, y).$$

We define an equivalent relation $\sim$ on $\mathbb{R} \times [0, 1]$ such that

$$\begin{cases} (x, 0) \sim (x', 0) & \text{for } x, x' \in \mathbb{R} \\ (x, y) \sim (x + 1, y) & \text{for } 0 < y < 1, \\ (x, 1) \sim (x', 1) & \text{for } x, x' \in \mathbb{R}. \end{cases}$$

Then $\mathbb{R} \times [0, 1]/_\sim$ is a sphere and g descends to an orientation and area preserving diffeomorphism f of the sphere that has exactly two fixed points. Note the facts that every maximal identity isotopy I fixes both fixed points of f, that the rotation number of I at each fixed point is an integer, and that the sum of the rotation numbers of I at both fixed point is 1. So, there does not exist any maximal isotopy I such that for all $z \in \mathrm{Fix}(I)$,

$$-1 < \rho(I, z) < 1.$$

4.4 Appendix: Construct a transverse foliation from the generating function

Let f be a diffeomorphism of $\mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^2$ function, we call g a *generating function* of f if $\partial^2_{12} g < 1$, and if

$$f(x, y) = (X, Y) \Leftrightarrow \begin{cases} X - x = & \partial_2 g(X, y), \\ Y - y = & -\partial_1 g(X, y). \end{cases}$$

Every $\mathcal{C}^2$ function $g : \mathbb{R}^2 \to \mathbb{R}$ satisfiying $\partial^2_{12} g \le c < 1$ defines a diffeomorphism f of $\mathbb{R}^2$ by the previous equations, and every area preserving diffeomorphism f of $\mathbb{R}^2$ satisfying $0 < \varepsilon \le \partial_1(p_1 \circ f) \le M < \infty$ can be generated by a generating function, where p_1 is the projection onto the first factor. Moreover, the Jacobian matrix J_f of f is equal to

$$\frac{1}{1 - \partial^2_{12} g(X, y)} \begin{pmatrix} 1 & \partial^2_{22} g(X, y) \\ -\partial^2_{11} g(X, y) & -\partial^2_{11} g(X, y)\partial^2_{22} g(X, y) + (1 - \partial^2_{12} g(X, y))^2 \end{pmatrix}.$$

Since $\det J_f = 1$, the diffeomorphism f is orientation and area preserving. A point (x, y) is a fixed point of f if and only if it is a critical point of g. We can naturally define an identity isotopy $I = (f_t)_{t \in [0,1]}$ of f such that f_t is generated by tg. Precisely, the diffeomorphisms f_t are defined by the following equations:

$$f_t(x, y) = (X^t, Y^t) \Leftrightarrow \begin{cases} X^t - x = & t\partial_2 g(X^t, y), \\ Y^t - y = & -t\partial_1 g(X^t, y). \end{cases}$$

In this section, we suppose that f is a diffeomorphism of $\mathbb{R}^2$, and that g is a generating function of f. We will construct a transverse foliation of I. More precisely, denote by $\mathcal{F}$ the foliation whose leaves are the integral curves of the gradient vector field $(x, y) \mapsto (\partial_1 g(x, y), \partial_2 g(x, y))$ of g, we will prove the following result:

Theorem 4.33. *The foliation $\mathcal{F}$ is a transverse foliation of I.*

Proof. We will prove the theorem by constructing an identity isotopy I' of f that is homotopic to I relatively to $\mathrm{Fix}(f)$ and satisfies that for every $z \in \mathbb{R}^2 \setminus \mathrm{Fix}(f)$, the trajectory of z along I' is positively transverse to $\mathcal{F}$.

We define $I' = (f'_t)_{t\in[0,1]}$ by the following equations:

$$f'_t(x, y) = \begin{cases} (x, y) + 2t(X - x, 0) & \text{for } 0 \le t \le 1/2, \\ (X, y) + (2t - 1)(0, Y - y) & \text{for } 1/2 \le t \le 1, \end{cases}$$

where $(X, Y) = f(x, y)$.

Lemma 4.34. *One can verify that I' is an identity isotopy of f.*

Proof. We know that $\partial_1 X(x, y) = 1/(1 - \partial^2_{12} g(X, y)) > 0$. By computing the determinant of the Jacobian matrix of f'_t, we know that $\det J_{f'_t} > 0$ for every $t \in [0, 1]$. Indeed, for $t \in [0, 1/2]$,

$$\det J_{f'_t} = \det \begin{pmatrix} 1 + 2t(\partial_1 X - 1)) & 2t\partial_2 X \\ 0 & 1 \end{pmatrix} = 2t\partial_1 X + (1 - 2t) > 0;$$

for $t \in [1/2, 1]$,

$$\begin{aligned} \det J_{f'_t} &= \det \begin{pmatrix} \partial_1 X & \partial_2 X \\ (2t - 1)\partial_1 Y & (2 - 2t) + (2t - 1)\partial_2 Y \end{pmatrix} \\ &= (2t - 1) \det J_f + (2 - 2t)\partial_1 X > 0. \end{aligned}$$

To prove that I' is an isotopy, we only need to check that f'_t is a bijection for every $t \in (0, 1)$.

For $t \in (0, \frac{1}{2})$, write $f_t(x, y) = (\varphi_{t,y}(x), y)$. One deduces

$$\frac{\partial}{\partial x}\varphi_{t,y}(x) = 2t\partial_1 X(x, y) + (1 - 2t) > 1 - 2t > 0.$$

So, f'_t is a surjection. Now, we will prove f'_t is an injection. Suppose that $f'_t(x, y) = f'_t(x', y')$, and write $(X', Y') = f'(x', y')$. One knows $y = y'$ and

$$x + 2t(X - x) = x' + 2t(X' - x').$$

So, one knows

$$X - (1 - 2t)\partial_2 g(X, y) = X' - (1 - 2t)\partial_2 g(X', y),$$

and deduces

$$(X - X') - (1 - 2t)\partial^2_{12} g(\xi, y)(X - X') = 0,$$

where ξ is a real number between X and X'. So, one knows $X = X'$. By definition of the generating function, one deduces $(x, y) = (x', y')$. Therefore, f'_t is injective.

For $t \in [\frac{1}{2}, 1)$, write $\psi_{t,X}(y) = y + (2t-1)(Y-y)$. One deduces

$$\frac{\partial}{\partial y}\psi_{t,X}(y) = 1 - (2t-1)\partial^2_{12}g(X,y) > 2 - 2t > 0.$$

So, f'_t is a surjection. Now, we will prove f'_t is an injection. Suppose that $f'_t(x,y) = f'_t(x',y')$, and write $(X',Y') = f'(x',y')$. One knows $X = X'$ and

$$y + (2t-1)(Y-y) = y' + (2t-1)(Y'-y').$$

So, one knows

$$(y-y') - (2t-1)(\partial_1 g(X,y) - \partial_1 g(X,y')) = 0,$$

and then deduces

$$(y-y') - (2t-1)\partial^2_{12}g(X,\eta)(y-y'),$$

where η is a real number between y and y'. So, one knows $y = y'$, and then deduces $(x,y) = (x',y')$. We conclude that f'_t is an injection. $\square$

By definition, we know $\mathrm{Fix}(I_0) = \mathrm{Fix}(I') = \mathrm{Fix}(f)$. If $\mathrm{Fix}(f)$ is empty or contains more than one point, we know that $(\mathrm{Fix}(f), I_0) \sim (\mathrm{Fix}(f), I')$, and hence a transverse foliation of I' is also a transverse foliation of I_0; if $\mathrm{Fix}(f)$ is reduced to one point, we can deduce the same result by the following lemma and the fact that $\pi_1(\mathrm{homeo}_0(\mathbb{R}^2, 0)) \cong \mathbb{Z}$.

Lemma 4.35. *If 0 is an isolated fixed point of f, one can deduce that $\rho(I,0) = \rho(I',0) \in [-1,1]$.*

Proof. Let $\theta : [0,1] \to \mathbb{R}$ and $\theta' : [0,1] \to \mathbb{R}$ be the continuous functions that satisfies $\theta(0) = \theta'(0) = 0$ and

$$\frac{J_{f_t}(0)\begin{pmatrix}1\\0\end{pmatrix}}{\|J_{f_t}(0)\begin{pmatrix}1\\0\end{pmatrix}\|} = \begin{pmatrix}\cos\theta(t)\\ \sin\theta(t)\end{pmatrix}, \quad \frac{J_{f'_t}(0)\begin{pmatrix}1\\0\end{pmatrix}}{\|J_{f'_t}(0)\begin{pmatrix}1\\0\end{pmatrix}\|} = \begin{pmatrix}\cos\theta'(t)\\ \sin\theta'(t)\end{pmatrix}.$$

To simplify the notations, we write

$$\mathrm{Hess}(g)(0) = \begin{pmatrix}\varrho, \sigma\\ \sigma, \tau\end{pmatrix}.$$

One knows

$$J_{f_t}(0)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{1-t\sigma}\begin{pmatrix}1\\ -t\varrho\end{pmatrix}.$$

We know $1 - t\sigma > 0$ for all $t \in [0,1]$, so $\theta(t)$ belongs to $(-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t \in [0,1]$.

For $t \in [0, \frac{1}{2}]$, one knows

$$J_{f'_t}(0)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}(1-2t) + 2t\partial_1 X(0,0)\\ 0\end{pmatrix}.$$

We know $(1-2t) + 2t\partial_1 X(0,0) > 0$, so $\theta'(t)$ is equal to 0 for all $t \in [0, \frac{1}{2}]$.

For $t \in [\frac{1}{2}, 1]$, one knows

$$J_{f'_t}(0)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\partial_1 X(0,0)\\ (2t-1)\partial_1 Y(0,0)\end{pmatrix}.$$

We know $\partial_1 X(0,0) > 0$, so $\theta'(t)$ belongs to $(-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t \in [\frac{1}{2}, 1]$.

Therefore, one deduces $\theta(1) = \theta'(1) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and hence $\rho(I, 0) = \rho(I', 0) \in [-1, 1]$. $\square$

Lemma 4.36. *For every $z = (x, y)$ that is not a fixed point of f, the path $\gamma_z : t \mapsto f'_t(x, y)$ is positively transverse to $\mathcal{F}$.*

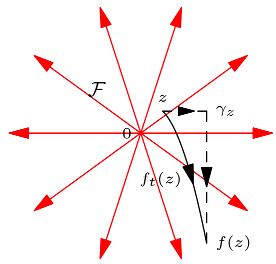


Figure 4.4: The dynamics and foliation generated by $g(x, y) = x^2 + y^2$

Proof. For $t \in [0, 1/2]$,

$$\begin{aligned} & \det \begin{pmatrix} 2(X-x) & \partial_1 g(f'_t(x,y)) \\ 0 & \partial_2 g(f'_t(x,y)) \end{pmatrix} \\ = \; & 2(X-x)\partial_2 g(f'_t(x,y)) \\ = \; & 2(X-x)\partial_2 g(2tX + (1-2t)x, y) \\ = \; & 2(X-x)\partial_2 g(X,y) + (2t-1)(X-x)\partial^2_{12} g(\xi, y)] \\ = \; & 2(X-x)^2[1-(1-2t)\partial^2_{12} g(\xi,y)] \geq 0 \end{aligned}$$

where ξ is a real number between x and X, and the inequality is strict if $X \neq x$.

For $t \in [1/2, 1]$,

$$\begin{aligned} & \det \begin{pmatrix} 0 & \partial_1 g(f'_t(x,y)) \\ 2(Y-y) & \partial_2 g(f'_t(x,y)) \end{pmatrix} \\ = \; & -2(Y-y)\partial_1 g(f'_t(x,y)) \\ = \; & -2(Y-y)\partial_1 g(X, (2-2t)y + (2t-1)Y) \\ = \; & -2(Y-y)[\partial_1 g(X,y) + (2t-1)(Y-y)\partial^2_{12} g(X,\eta)] \\ = \; & 2(Y-y)^2[1-(2t-1)\partial^2_{12} g(X,\eta)] \geq 0 \end{aligned}$$

where η is a real number between y and Y, and the inequality is strict if $Y \neq y$.

Since z is not a fixed point, either $X \neq x$ or $Y \neq y$. If both of the inequalities are satisfied, γ_z intersects $\mathcal{F}$ positively transversely; if $X \neq x$ and $Y = y$, $\gamma_z|_{t\in[0,\frac{1}{2}]}$ intersects $\mathcal{F}$ positively transversely, and $\gamma_z|_{t\in[\frac{1}{2},1]}$ is reduced to a point; if $X = x$ and $Y \neq y$, $\gamma_z|_{t\in[0,\frac{1}{2}]}$ is reduced to a point, and $\gamma_z|_{t\in[\frac{1}{2},1]}$ intersects $\mathcal{F}$ positively transversely. $\square$

$\square$

Chapter 5

A generalization of the local Poincaré-Birkhoff theorem

5.1 Proof of the main results

Let f be an orientation preserving homeomorphism of the annulus $\mathbb{T}^1 \times [0, +\infty)$. Let $C_0 = \mathbb{T}^1 \times \{0\}$, C_1 be an essential loop in $\mathbb{T}^1 \times (0, +\infty)$ that projects injectively to the first factor, and $C_2 = f(C_1)$. We denote by A_i the closed annulus bounded by C_0 and C_i, and write $\mathrm{Int}(A_i) = A_i \setminus C_i$ for $i = 1, 2$. Let $\pi : \mathbb{R} \times [0, \infty) \to \mathbb{T}^1 \times [0, \infty)$ be the universal cover, and $\widetilde{f}$ be a lift of f. Let $p_1 : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ be the projection onto the first factor. We denote by $\widetilde{C}_i$, $\widetilde{A}_i$ and $\mathrm{Int}(\widetilde{A}_i)$ the pre-images of C_i for $i = 0, 1, 2$, A_i and $\mathrm{Int}(A_i)$ for $i = 1, 2$ respectively. We denote by $\mathrm{Fix}_*(f)$ the set of fixed points of f lifted to fixed points of $\widetilde{f}$. We will prove the following two results in this section.

Theorem 5.1 (Theorem 1.12). *If* $\mathrm{Fix}_*(f) \cap C_0 = \{z_0\}$*, and if* $p_1(\widetilde{f}(\widetilde{z}) - \widetilde{z}) p_1(\widetilde{f}(\widetilde{z}') - \widetilde{z}') < 0$ *for all* $\widetilde{z} \in \widetilde{C}_0 \setminus \pi^{-1}(z_0)$ *and* $\widetilde{z}' \in \widetilde{C}_1$*, then we are in one of the following cases:*
i) there exists a fixed point of f *in the interior of* A_1 *lifted to fixed points of* $\widetilde{f}$*;*
ii) there exists an essential loop γ *in* A_1 *that does not intersect* $C_0 \setminus \{z_0\}$ *and satisfies* $\gamma \cap f(\gamma) \subset \{z_0\}$*.*

Corollary 5.2 (Corollary 1.13). *If* A_1 *does not contain the positive or the negative orbit of any wandering open set and if there exists* $\widetilde{z} \in \widetilde{C}_0$ *such that* $p_1(\widetilde{f}(\widetilde{z}) - \widetilde{z}) p_1(\widetilde{f}(\widetilde{z}') - \widetilde{z}') < 0$ *for all* $\widetilde{z}' \in \widetilde{C}_1$*, then there exists a fixed point of* f *in the interior of* A_1 *lifted to fixed points of* $\widetilde{f}$*.*

Before proving the theorem, we will first prove the corollary:

Proof of Corollary 1.13. One has to consider two cases:
- there exists $\widetilde{z} \in \widetilde{C}_0$ such that $p_1(\widetilde{f}(\widetilde{z}) - \widetilde{z}) < 0$, and $p_1(\widetilde{f}(\widetilde{z}') - \widetilde{z}') > 0$ for all $\widetilde{z}' \in \widetilde{C}_1$;
- there exists $\widetilde{z} \in \widetilde{C}_0$ such that $p_1(\widetilde{f}(\widetilde{z}) - \widetilde{z}) > 0$, and $p_1(\widetilde{f}(\widetilde{z}') - \widetilde{z}') < 0$ for all $\widetilde{z}' \in \widetilde{C}_1$;

We only study the first case, the second one can be treated similarly.

Let $l \subset \mathbb{R} \times \{0\}$ be a maximal interval such that $p_1 \widetilde{f}(x, 0) < x$ for all $x \in l$. Shrinking the complement of the interval $\pi(l)$ in C_0 to a point, we get an annulus $\breve{A}$ and a homeomorphism $\breve{f}$ of $\breve{A}$. Let $\breve{A}_1 \subset \breve{A}$ be the quotient space of A_1. Let $\varepsilon > 0$ be a small number such that $f(C_1) \subset \mathbb{T}^1 \times (\varepsilon, \infty)$, and h be a homeomorphism between $\breve{A}$ and A such that h is equal the identity in $\mathbb{T}^1 \times [\varepsilon, \infty)$. Let $f' = h \circ \breve{f} \circ h^{-1}$, and $\widetilde{f}'$ be the lift of f' that is equal to $\widetilde{f}$ in $\mathbb{R} \times [\varepsilon, \infty)$. Then, f' satisfies the condition of the previous theorem but does not satisfy ii). Consequently, there exists a fixed point of f' in the interior of $\breve{A}_1$ lifted to fixed point of $\widetilde{f}'$, and so there there exists a fixed point of f in the interior of A_1 lifted to fixed points of $\widetilde{f}$. □

Now we begin the proof of Theorem 1.12. One has to consider two cases:
- $p_1(\widetilde{f}(\widetilde{z}) - \widetilde{z}) < 0$ for all $\widetilde{z} \in \widetilde{C}_0 \setminus \pi^{-1}(z_0)$, and $p_1(\widetilde{f}(\widetilde{z}') - \widetilde{z}') > 0$ for all $\widetilde{z}' \in \widetilde{C}_1$;
- $p_1(\widetilde{f}(\widetilde{z}) - \widetilde{z}) > 0$ for all $\widetilde{z} \in \widetilde{C}_0 \setminus \pi^{-1}(z_0)$, and $p_1(\widetilde{f}(\widetilde{z}') - \widetilde{z}') < 0$ for all $\widetilde{z}' \in \widetilde{C}_1$;

We only study the first case, and the second one can be treated similarly.

We construct an open annulus $A' = \mathbb{T}^1 \times \mathbb{R}$ by pasting two copies of the annulus A at the lower boundary C_0, and a homeomorphism on A' that coincides with f on each copy. Let σ be the natural involution. To simplify the notations, we still denote by f this homeomorphism. Let $\pi' : \mathbb{R}^2 \to A'$ be the universal cover, and $T : (x, y) \mapsto (x + 1, y)$ be the translation on $\mathbb{R}^2$. We still denote by $\widetilde{f}$ the lift of f. Let $A'_1 = A_1 \cup \sigma(A_1)$ and $\widetilde{A}'_1 = \pi^{-1}(A_1)$. Let $C_0^* = C_0 \setminus \{z_0\}$ and $\widetilde{C}_0^* = \pi'^{-1}(C_0^*)$. We denote by γ^{-1} the path $t \mapsto \gamma(1 - t)$ for any path γ defined on $[0, 1]$.

Let $g : W \to R^2$ be a continuous map defined on an open set W, and $\gamma : [0, 1] \to W$ be a path that does not intersect the fixed points set of g, we define *the index of g along γ* by

$$i(g, \gamma) = \int_{\gamma'} d\theta,$$

where $\gamma' : t \mapsto f(\gamma(t)) - \gamma(t)$ is a path in $\mathbb{R}^2 \setminus \{0\}$, and $d\theta = \frac{1}{2\pi} \frac{xdy - ydx}{x^2 + y^2}$ is the polar form.

Remark 5.3. In particular, if γ is a simple loop, and if g is defined on the domain D bounded by γ and has only finitely many fixed points in the domain D bounded by γ, then

$$i(g, \gamma) = \pm \sum_{z \in \mathrm{Fix}(g) \cap D} i(g, z),$$

where the sign depends on the orientation of γ.

We will prove the theorem by contradiction. Suppose that we are in neither of the two cases of Theorem 1.12. On one hand, $\mathrm{Fix}_*(f) \cap A_1$ is reduced to z_0, and we can prove that the Lefschetz index $i(\widetilde{f}, \widetilde{z}_0)$ is zero for $\widetilde{z}_0 \in \pi^{-1}(z_0)$. On the other hand, following the approach of [LCW10] which used the notion of "positive path" and was inspired by Birkhoff's paper [Bir26], we will construct a loop $\Gamma \subset (\widetilde{A}'_1)$ such that $i(\widetilde{f}, \Gamma) = -2$. We will see that Theorem 1.12 is a corollary of the following Lemma 5.4 - Lemma 5.10.

Lemma 5.4. *If* $\mathrm{Fix}_*(f) \cap \mathrm{Int}(A_1)$ *is reduced to* z_0*, then* $i(\widetilde{f}, \widetilde{z}_0) = 0$ *for every* $\widetilde{z}_0 \in \pi'^{-1}(z_0)$,

Proof. Let γ_0 be a path in A'_1 joining $\sigma(C_1)$ to C_1 that does not intersect $\{z_0\}$. Lift γ_0 to an arc γ joining $\sigma(\widetilde{C}_1)$ to $\widetilde{C}_1$. Let γ' be the segment of $\widetilde{C}_1$ joining $\gamma(1)$ to $T(\gamma(1))$, then $\Gamma = \gamma\gamma' T(\gamma)^{-1} \sigma(\gamma')^{-1}$ is a loop, and the domains bounded by Γ contains a unique lift $\widetilde{z}_0$ of z_0. We have

$$i(\widetilde{f}, \widetilde{z}_0) = -i(\widetilde{f}, \Gamma) = -i(\widetilde{f}, \gamma) - 0 + i(\widetilde{f}, \gamma) + 0 = 0,$$

because $\widetilde{f}$ commutes with T and σ. □

Let G be a homeomorphism of a topological space X. We say that a path $\gamma : [0, 1] \to X$ is a *positive path* if for every $t, t' \in [0, 1]$,

$$t' \geq t \Rightarrow G(\gamma(t')) \neq \gamma(t).$$

Note that if γ is a positive path of G, then $G(\gamma)^{-1}$ is a positive path of G^{-1}. Let A and B be two disjoint subset of X, we say that a path $\gamma : [0, 1] \to X$ *joins A to B* if $\gamma(0) \in A$, $\gamma(1) \in B$, and $\gamma(t) \notin A \cup B$ for all $t \in (0, 1)$.

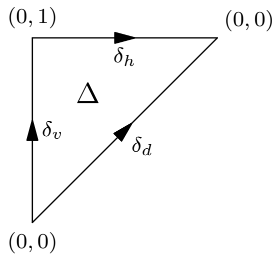
Lemma 5.5. *If γ is a positive path of $\widetilde{f}$ joining $\widetilde{C}_0^*$ to $\widetilde{C}_1$ such that $\widetilde{f}(\gamma(1)) \notin \mathrm{Int}(\widetilde{A}_1)$, if γ' is a positive path of $\widetilde{f}$ joining $\widetilde{C}_1$ to $\widetilde{C}_0^*$ such that $\gamma'(0) \notin \mathrm{Int}(\widetilde{A}_2)$ is at the right of $\gamma(1)$, and if γ'' is the segment of $\widetilde{C}_1$ joining $\gamma(1)$ to $\gamma'(0)$, then*

$$i(\widetilde{f}, \Gamma) = -2,$$

where Γ is the loop $\gamma\gamma''\gamma'\sigma(\gamma')^{-1}\sigma(\gamma'')^{-1}\sigma(\gamma)^{-1}$.

Proof. By a suitable conjugation that preserves the vertical lines, we can suppose $A_1 = \mathbb{T}^1 \times [0,1]$.

Let us consider the simplex $\Delta = \{(t,t') : 0 \leq t \leq t' \leq 1\}$. Let δ_v, δ_h and δ_d be the



paths along the edges of Δ joining $(0,0)$ to (0,1), joining $(0,1)$ to $(1,1)$ and joining $(0,0)$ to $(1,1)$ respectively. The application $\Phi(t,t') = \widetilde{f}(\gamma(t')) - \gamma(t)$ does not vanish on Δ, and hence

$$\int_{\Phi\circ\delta_d} d\theta = \int_{\Phi\circ\delta_v} d\theta + \int_{\Phi\circ\delta_h} d\theta.$$

The first integral is just $i(\widetilde{f}, \gamma)$. The second coordinate of the curves $\Phi \circ \delta_v$ and $\Phi \circ \delta_h$ are both non-negative, the initial point $\Phi(0,0) \subset (-\infty, 0) \times \{0\}$, the end point $\phi(1,1) \subset (0,\infty) \times [0,\infty)$. One deduces that

$$i(\widetilde{f}, \gamma) \in [-\frac{1}{2}, -\frac{1}{4}).$$

The image of the path $t \mapsto \widetilde{f}(\gamma''(t)) - \gamma''(t)$ is contained in $(0,\infty) \times \mathbb{R}$, so

$$i(\widetilde{f}, \gamma'') \in (-\frac{1}{4}, \frac{1}{4}).$$

Now, it remains to compute $i(\widetilde{f}, \gamma')$, which is harder than the computations of the indices of the previous two paths. Let $T_r(x,y) = (x+r, y)$. We fix a large positive number r_0 such that

$$p_1(\widetilde{f}(T_{r_0}(\gamma'(t'))) - \gamma'(0)) > 0, \text{ for } t' \in [0,1].$$

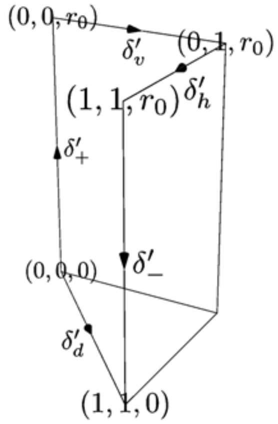
Let ε be a sufficiently small positive number such that

$$\min_{z \in \widetilde{C}_1} p_1(\widetilde{f}(z) - z) > r_0\varepsilon,$$

and that

$$\min_{t' \in [0,1]} \widetilde{f}(T_{r_0}(\gamma'(t'))) - \gamma'(0) > r_0\varepsilon.$$

We construct a continuous map $\Psi(t,t',r) = \widetilde{f}(T_r(\gamma'(t'))) - T_{tr(1+\varepsilon)}\gamma'(t)$ on $\Delta \times [0, r_0]$. Let us prove that Ψ does not vanish on the closed surface $(\Delta \times \{0\}) \cup \{t = 0\} \cup \{t' = 1\}$. Indeed, it does not vanish on $\Delta \times \{0\}$, because γ' is a positive path for $\widetilde{f}$. Suppose that



$t = t' = 0$, and note that $\Psi(0,0,r) = \widetilde{f}(T_r(\gamma'(0))) - \gamma'(0)$. Its image is contained in $(0,\infty)\times\mathbb{R}$, because the first coordinate satisfies

$$p_1(\Psi(0,0,r)) = p_1(\widetilde{f}(T_r(\gamma'(0))) - T_r(\gamma'(0))) + r > 0.$$

Suppose that $t = 0, t' \in (0,1)$, and note that $\Psi(0,t',r) = \widetilde{f}(T_r(\gamma'(t'))) - \gamma'(0)$ does not vanish, because $\gamma'(0) \notin \mathrm{Int}(\widetilde{A}_2)$, and $\widetilde{f}(T_r(\gamma'(t'))) \in \mathrm{Int}(\widetilde{A}_2)$ for $t' \in (0,1)$. Suppose that $t \in [0,1), t' = 1$, and note that $\Psi(t,1,r) = \widetilde{f}(T_r(\gamma'(1))) - T_{tr(1+\varepsilon)}\gamma'(t)$. Its image is contained in $\mathbb{R}\times(-\infty,0)$. Suppose that $t = t' = 1$, and note that $\Psi(1,1,r) = \widetilde{f}(T_r(\gamma'(1))) - T_{r+r\varepsilon}\gamma'(1)$. Its image is contained in $(-\infty,0)\times\{0\}$.

Let δ'_d be the segment joining $(0,0,0)$ to $(1,1,0)$, δ'_+ be the segment joining $(0,0,0)$ to $(0,0,r_0)$, δ'_v be the segment joining $(0,0,r_0)$ to $(0,1,r_0)$, δ'_h be the segment joining $(0,1,r_0)$ to $(1,1,r_0)$, and δ'_- be the segment joining $(1,1,r_0)$ to $(1,1,0)$. So,

$$i(\widetilde{f},\gamma') = \int_{\Psi\circ\delta'_d} d\theta = \int_{\Psi\circ(\delta'_+\delta'_v\delta'_h\delta'_-)} d\theta.$$

The image of $\Psi\circ\delta'_+$ is contained in $(0,\infty)\times\mathbb{R}$, the image of $\Psi\circ\delta'_v$ is contained in $(0,\infty)\times\mathbb{R}$, the image of $\Psi\circ\delta'_h$ is contained in $\mathbb{R}\times(-\infty,0)\cup(-\infty,0)\times\{0\}$, and the image of $\Psi\circ\delta_-$ is contained in $(-\infty,0)\times\{0\}$. Then

$$i(\widetilde{f},\gamma') \in (-\frac{3}{4},-\frac{1}{4}).$$

The images of both the initial point $\widetilde{f}(\gamma(0)) - \gamma(0)$ and the end point $\widetilde{f}(\gamma'(1)) - \gamma'(1)$ are contained in $(-\infty,0)\times\{0\}$, so $i(\widetilde{f},\gamma\gamma''\gamma')$ is an integer. Recall that $i(\widetilde{f},\gamma)$ is included in $[-\frac{1}{2},-\frac{1}{4})$, that $i(\widetilde{f},\gamma'')$ is included in $(-\frac{1}{4},\frac{1}{4})$, and that $i(\widetilde{f},\gamma')$ is included in $(-\frac{3}{4},-\frac{1}{4})$. So,

$$i(\widetilde{f},\gamma\gamma''\gamma') = i(\widetilde{f},\gamma) + i(\widetilde{f},\gamma'') + i(\widetilde{f},\gamma') \in (-\frac{3}{2},-\frac{1}{4}).$$

Therefore

$$i(\widetilde{f},\gamma\gamma''\gamma') = -1,$$

and

$$i(\widetilde{f},\Gamma) = i(\widetilde{f},\gamma\gamma''\gamma') + i(\widetilde{f},\sigma(\gamma\gamma''\gamma')^{-1}) = -2.$$

□

Lemma 5.6. *If there exists a positive path γ of $\widetilde{f}$ in $\widetilde{A}_1$ starting from $\widetilde{C}_0^*$ such that $\gamma|_{[0,1)} \subset \mathrm{Int}(\widetilde{A}_1)$ and $\widetilde{f}(\gamma) \not\subset \mathrm{Int}(\widetilde{A}_1)$, there exists a positive path γ' of $\widetilde{f}$ joining $\widetilde{C}_0^*$ to $\widetilde{C}_1$ such that $\widetilde{f}(\gamma'(1)) \notin \mathrm{Int}(\widetilde{A}_1)$.*

Proof. Taking a sub-path if necessary, we can suppose that $\gamma(t)$ is in the interior of $\widetilde{A}_1$ for all $t \in (0,1)$ and $\widetilde{f}(\gamma(1)) \notin \mathrm{Int}(\widetilde{A}_1)$. If $\gamma(1) \in \widetilde{C}_1$, the path γ satisfies the lemma. Suppose $\gamma(1) \notin \widetilde{C}_1$. Let δ be an arc joining $\widetilde{f}(\gamma(1))$ to $\widetilde{C}_2$ such that δ does not intersect $\widetilde{A}_1$ if $\widetilde{f}(\gamma(1)) \notin \widetilde{C}_1$, and that δ intersects $\widetilde{A}_1$ at only one point $\widetilde{f}(\gamma(1))$ if $\widetilde{f}(\gamma(1)) \in \widetilde{C}_1$. Then $\gamma' = \gamma \widetilde{f}^{-1}(\delta)$ is a positive path of $\widetilde{f}$ joining $\widetilde{C}_0^*$ to $\widetilde{C}_1$ such that $\widetilde{f}(\gamma'(1)) \notin \mathrm{Int}(\widetilde{A}_1)$. □

We define a *free disk chain* of $\widetilde{f}$ in $\widetilde{A}_1$ to be a the sequence $(\sigma_i)_{0\leq i\leq n}$ of closed topological disks σ_i of $\widetilde{A}$ whose interiors are pairwise disjoint and such that $\widetilde{f}(\sigma_i) \cap \sigma_i = \emptyset$ for all $0 \leq i \leq n$, and $\widetilde{f}(\sigma_{i-1}) \cap \sigma_i \neq \emptyset$ for all $1 \leq i \leq n$.

Lemma 5.7. *If there exists a free disk chain $(\sigma_i)_{0\leq i\leq n}$ of $\widetilde{f}$ in $\widetilde{A}_1$ such that $\sigma_0 \cap \widetilde{C}_0^* \neq \emptyset$, and $\widetilde{f}(\sigma_n) \not\subseteq \mathrm{Int}(\widetilde{A}_1)$, there exists a positive path γ of $\widetilde{f}$ starting from $\widetilde{C}_0^*$ such that $\gamma|_{[0,1)} \subset \mathrm{Int}(\widetilde{A}_1)$ and $\widetilde{f}(\gamma) \not\subseteq \mathrm{Int}(\widetilde{A}_1)$.*

Proof. Let $m = \min\{n :$ there exists a free disk chain $(\sigma_i)_{0\leq i\leq n}$ in $\widetilde{A}_1$ such that $\sigma_0 \cap \widetilde{C}_0^* \neq \emptyset$, and $\widetilde{f}(\sigma_n) \not\subseteq \mathrm{Int}(\widetilde{A}_1)\}$, and $(\sigma_i)_{0\leq i\leq m}$ be one of such free disk chain. We will show that

$$\widetilde{f}^k(\sigma_i) \cap \sigma_j = \emptyset \text{ for } j > i+1 \text{ and } k \geq 1.$$

Otherwise, let $i_0 = \min\{i :$ there exist $j > i+1, k \geq 1,$ such that $\widetilde{f}^k(\sigma_i) \cap \sigma_j \neq \emptyset\}$, $j_0 = \max\{j :$ there exists $k \geq 1,$ such that $\widetilde{f}^k(\sigma_{i_0}) \cap \sigma_j \neq \emptyset\}$, and $k_0 = \min\{k : \widetilde{f}^k(\sigma_{i_0}) \cap \sigma_{j_0} \neq \emptyset\}$. Then we get a free disk chain

$$(\widetilde{f}^{k_0-1}(\sigma_0), \cdots, \widetilde{f}^{k_0-1}(\sigma_{i_0}), \sigma_{j_0}, \cdots, \sigma_m),$$

and $\widetilde{f}^{k_0-1}(\sigma_0) \cap \widetilde{C}_0^* \neq \emptyset$. This contradicts the minimality of m.

The set $\cup_{0\leq i\leq m} \widetilde{f}^{m-i}(\sigma_i)$ is connected and intersects $\widetilde{C}_0^*$. Moreover, $\widetilde{f}(\sigma_m) \not\subseteq \mathrm{Int}(\widetilde{A}_1)$. Choose a point $z_0 \in \widetilde{f}^m(\sigma_0) \cap \widetilde{C}_0^*$, a point $z_{m+1} \in \sigma_m$ such that $\widetilde{f}(z_{m+1}) \notin \mathrm{Int}(\widetilde{A}_1)$, and a point $z_i \in \widetilde{f}^{m-i+1}(\sigma_{i-1}) \cap \widetilde{f}^{m-i}(\sigma_i)$ for $1 \leq i \leq m$. Choose an arc γ_i joining z_i to z_{i+1} such that $\gamma_i \setminus \{z_i, z_{i+1}\} \subset \mathrm{Int}(\widetilde{f}^{m-i}(\sigma_i))$ for $0 \leq i \leq m$. The path $\gamma = \gamma_0 \cdots \gamma_m$ is a path starting from $\gamma(0) \in \widetilde{C}_0^*$, and satisfying $\widetilde{f}(\gamma) \not\subseteq \mathrm{Int}(\widetilde{A}_1)$. By choosing suitable γ_i for $i = 0, \cdots, m$, we can suppose that $\gamma|_{[0,1)}$ does not intersect $\widetilde{C}_1$.

We will prove that γ is a positive path. Indeed, we will prove that $\widetilde{f}(\gamma_j) \cap \gamma_i = \emptyset$ for $0 \leq i \leq j \leq m$. By the construction, we know that $\gamma_i \subset \widetilde{f}^{m-i}(\sigma_i)$, and σ_i is a free disk, so $\widetilde{f}(\gamma_i) \cap \gamma_i = \emptyset$ for $i = 0, \cdots, m$. For $0 \leq i+1 < j \leq m$, one deduces that $\widetilde{f}(\gamma_j) \cap \gamma_i \subset \widetilde{f}^{m-j+1}(\sigma_j \cap \widetilde{f}^{j-i-1}(\sigma_i))$, and knows that $\sigma_j \cap \widetilde{f}^{j-i-1}(\sigma_i) = \emptyset$. So, $\widetilde{f}(\gamma_j) \cap \gamma_i = \emptyset$. We will see also that $\widetilde{f}(\gamma_j) \cap \gamma_{j-1} = \emptyset$ for $j = 1, \cdots, m$. Indeed, $\widetilde{f}(z_0) \neq z_0$, $\widetilde{f}(z_j) \notin \gamma_0 \cdots \gamma_{j-1}$ and $\widetilde{f}(\gamma_j \setminus \{z_j, z_{j+1}\}) \cap (\gamma_{j-1} \setminus \{z_{j-1}, z_j\}) \subset \mathrm{Int}(\widetilde{f}^{m-j+1}(\sigma_j)) \cap \mathrm{Int}(\widetilde{f}^{m-j+1}(\sigma_{j-1})) = \emptyset$, for $1 \leq j \leq m$. □

Replacing $\widetilde{f}$ with $\widetilde{f}^{-1}$ and $\widetilde{C}_2$ with $\widetilde{C}_1$, we get similarly

Lemma 5.8. *If there exists a free disk chain $(\sigma_i)_{0\leq i\leq n}$ in $\widetilde{A}_2$ of $\widetilde{f}^{-1}$ such that $\sigma_0 \cap \widetilde{C}_0^* \neq \emptyset$ and $\widetilde{f}^{-1}(\sigma_n) \not\subseteq \mathrm{Int}(\widetilde{A}_2)$, then there exists a positive path γ' of $\widetilde{f}^{-1}$ joining $\widetilde{C}_0$ to $\widetilde{C}_2$, such that $\widetilde{f}^{-1}(\gamma'(1)) \notin \mathrm{Int}(\widetilde{A}_2)$. Moreover, $\gamma'' = (\widetilde{f}^{-1}(\gamma'))^{-1}$ is a positive path of $\widetilde{f}$ joining $\widetilde{C}_1$ to $\widetilde{C}_0^*$, and $\gamma''(0) \notin \mathrm{Int}(\widetilde{A}_2)$.*

Lemma 5.9. *If we are in neither of the two cases i) and ii) of Theorem 1.12, then there exists a free disk chain satisfying the conditions of Lemma 5.7.*

Proof. Let $W = A_1 \setminus \mathrm{Fix}_*(f) = A_1 \setminus \{z_0\}$. Consider a brick decomposition $\mathcal{B} = (\sigma_i)_{i\in\mathcal{I}}$ of W, such that the bricks are closed disks obtained as the closures of the connected components of the complement in W of a locally finite graph Σ whose vertices are locally the extremities of exactly 3 edges. Moreover, we can suppose that the bricks are free when lifted to $\widetilde{A}_1$ by decomposing the bricks into smaller ones.

Let $\mathcal{J} \subset \mathcal{I}$ be the set of indices j such that there exists a sequence $(\sigma_{i_k})_{0\leq k\leq l}$ with $\sigma_{i_0} \cap C_0^* \neq \emptyset$, $\sigma_{i_l} = \sigma_j$, and $f(\sigma_{i_k}) \cap \sigma_{i_{k+1}} \neq \emptyset$. Then $X = \cup_{j\in\mathcal{J}}\sigma_j$ is a subsurface with boundary of W that contains a neighborhood of C_0^* in W.

We assert that there exists $j \in \mathcal{J}$ such that $f(\sigma_j) \not\subset \mathrm{Int}(A_1)$. We prove it by contradiction. Suppose that $f(\sigma_j) \subset \mathrm{Int}(A_1)$ for all $j \in \mathcal{J}$. Then $f(X) \subset \mathrm{Int}(A_1)$. We write $A^* = (\mathbb{T}^1 \times [0,\infty)) \setminus \{z_0\}$, then $f(X) \subset \mathrm{Int}_{A^*}(X)$. The boundary ∂X of X in A^* does not intersect $f(\partial X)$, and a component γ of ∂X is either a simple loop, or becomes a simple loop when one adds z_0. Since C_0^* and $A^* \setminus A_1$ are in different component of $A^* \setminus \partial X$, ∂X has either a component that is an essential loop or a component that becomes a simple loop when one adds z_0. This means we are in the second case of Theorem 1.12.

We choose suitable lifts of the disks $(\sigma_j)_{j\in\mathcal{J}}$, and can find a free disk chain of $\widetilde{f}$ satisfying the conditions of Lemma 5.7. □

Similarly, by replacing f with f^{-1}, we have the following lemma:

Lemma 5.10. *If we are in neither of the two cases i) and ii) of Theorem 1.12, then there exists a free disk chain satisfying the conditions of Lemma 5.8.*

Now we will give a proof of Theorem 1.12.

Proof of Theorem 1.12. We will prove the theorem by contradiction. Suppose that we are in neither of the two cases i) and ii) of Theorem 1.12. By Lemma 5.6, Lemma 5.7, and Lemma 5.9, one deduces that there exists a positive path γ of $\widetilde{f}$ joining $\widetilde{C}_0^*$ to $\widetilde{C}_1$ such that $\widetilde{f}(\gamma(1)) \notin \mathrm{Int}(\widetilde{A}_1)$. By Lemma 5.8 and Lemma 5.10, one deduces that there exists a positive path γ' of $\widetilde{f}$ joining $\widetilde{C}_1$ to $\widetilde{C}_0^*$ such that $\gamma'(0) \notin \mathrm{Int}(\widetilde{A}_2)$. Moreover, by considering a translation of γ', we can suppose that $\gamma'(0)$ is at the right of $\gamma(1)$. So, by Lemma 5.5, one deduces that there exists a loop Γ in $\widetilde{A}_1'$ such that $i(\widetilde{f},\Gamma) = -2$. But by Lemma 5.4, one knows that $\widetilde{f}$ does not have any fixed point with non-zero index, we get a contradiction. □

5.2 An application to the local dynamics of homeomophisms

In this section, suppose that $f : (W, z_0) \to (W', z_0)$ is a local homeomorphism at z_0 such that there exists a neighborhood of z_0 that does not contain the positive or the negative orbit of any wandering open set, and that $I = (f_t)_{t\in[0,1]}$ is a local isotopy of f that has a positive (resp. negative) rotation type. We suppose also that either f can be blown-up at z_0, or that z_0 is a non-accumulated indifferent point. Recall that in both case, we can define a rotation number $\rho(I, z_0)$. We say that a homeomorphism h of the circle is *right semi-stable* (resp. *left semi-stable*) if its lift $\widetilde{h}$ to $\mathbb{R}$ satisfies $\widetilde{h}(x) \geq x$ (resp. $\widetilde{h}(x) \leq x$) for all $x \in \mathbb{R}$, and the equality holds at some points. The aim of this section is to prove the following corollary:

Corollary 5.11 (Corollary 1.14)**.** *Under the previous assumptions, if $\rho(I, z_0) = 0$, the dynamics on the circle added when blowing-up is right semi-stable (resp. left semi-stable).*

Remark 5.12. When z_0 is a non-accumulated indifferent point, this result have been proven by Le Calvez [LC03] in a different way.

Proof of Corollary 1.14. Suppose that I has a positive rotation type, and that f can be blown-up at z_0. The proofs in the other cases is similar. To simplify the notations, we can also suppose that $z_0 = 0 \in \mathbb{R}^2$.

We will prove the proposition by contradiction. Let $\varphi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{T}^1 \times (-\infty, 0)$ be an orientation preserving homeomorphism such that $\varphi \circ f \circ \varphi^{-1}$ can be extended continuously to $\mathbb{T}^1 \times \{0\}$. Denote by h the extension homeomorphism on $\mathbb{T}^1 \times \{0\}$. We suppose that h is not right semi-stable, and will find a contradiction.

We will first show that 0 is an isolated fixed point of f by contradiction. Suppose that 0 is accumulated by fixed points of f. Since I has a positive rotation type, there exists a locally transverse foliation $\mathcal{F}$ of I such that 0 is a sink of $\mathcal{F}$. So, for a fixed point z of f that is sufficiently close to 0, the change of angular coordinate along the trajectory of z is a positive integer. Therefore, by definition of the local rotation set, we know that $\rho_s(I, 0)$ contains a positive integer or $+\infty$. On the other hand, we have $\rho(I, z_0) = 0$, which contradicts the assertion vii) of Proposition 2.20.

Let $U \subset W$ be a small neighborhood of 0 that contains neither the positive nor the negative orbit of any wandering open set. By choosing U sufficiently small, we can also suppose that f does not have any fixed point in U except 0. Using the same technique of the proof of Lemma 4.1, we can find an orientation homeomorphism φ' such that φ' is equal to φ in sufficiently small neighborhood of 0 and that there exists a circle in $\varphi'(U \setminus \{0\})$ such that $\varphi' \circ f \circ \varphi'^{-1}$ maps each point in the circle to the right. More precisely, let $\pi : \mathbb{R} \times (-\infty, 0] \to \mathbb{T}^1 \times (-\infty, 0]$ be the universal covering map, and $\widetilde{f}$ the lift of the extension of $\varphi' \circ f \circ \varphi'^{-1}$ (associated to the local isotopy $(\varphi' \circ f_t \circ \varphi'^{-1})_{t\in[0,1]}$). There exists $r > 0$ such that $\varphi'^{-1}(\mathbb{T}^1 \times [-r, 0)) \subset U$ and that

$$p_1(\widetilde{f}(x, -r)) - x > 0, \quad \text{for all } x \in \mathbb{R},$$

where p_1 is the projection onto the first factor. But by our assumption, h is not right semi-stable, so there exists $x_0 \in \mathbb{R}$ such that

$$p_1(\widetilde{f}(x_0, 0)) - x_0 < 0.$$

One deduces by Corollary 1.13 that $\varphi' \circ f \circ \varphi'^{-1}$ has a fixed point in $\varphi'(U \setminus \{0\})$, and hence there is a fixed point of f in $U \setminus \{0\}$. We get a contradiction. $\square$

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